Differential Equations

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- An equation that contains some derivatives of unknown function is called a **differential equation**.
- **Example:**

$$m\frac{d^2h}{dt^2} = -mg,$$

where m is the mass of the object, h is the height above the ground, g is the gravitational acceleration and h is the unknown function.

- If an equation involves the derivative of one variable with respect to another, the variables are called **dependent variable** and **independent variable** respectively.
- **Example:**

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

t is the independent variable and x is the dependent variable. a and k are called coefficients.

- A differential equation involving only ordinary derivatives with respect to single independent variable is an **ordinary differential equation.Example:**

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

- A differential equation involving partial derivatives with respect to more than independent variable is a **partial differential equation**. **Example:**

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$$

- The order of a differential equation is the order of the highest-order derivatives present.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$$

is second order differential equation.

- **Ordinary linear differential equation**

$$a_(x)\frac{d^ny}{dx^n} + a_{n-1}\frac{dy^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x),$$

- If the differential equation is no linear it called nonlinear.

Homework Problems

- Do problems 1-12, 13, and 15, page 5.

- A general form for nth order equation with x independent, y dependent:

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0$$

-

$$\frac{d^n y}{dx^n} = F(x, y, \frac{dy}{dx}, ..., \frac{d^{n-1} y}{dx^{n-1}})$$

satisfies the equation for all x in the interval I is called an explicit solution on I.

Explicit Solution

- A function ϕ that when substituted for y in equation

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0$$

or

$$\frac{d^n y}{dx^n} = F(x, y, \frac{dy}{dx}, ..., \frac{d^{n-1} y}{dx^{n-1}})$$

Examples

- Show that $\phi(x) = x^2 - \frac{1}{x}$ is an explicit solution to the linear equation
$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0.$$
- Show that $\psi(x) = x^3$ is not.
- Read examples 2 and 3 on page 7.

Implicit Solution

A relation $G(x, y) = 0$ is said to be an implicit solution to equation

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0.$$

- Show that

$$x + y + e^{xy} = 0$$

is an implicit solution to nonlinear equation.

- Read example 5, page 5.

Initial Value Problem

The nth order differential equation

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0,$$

with n initial conditions

$$y(x_0) = y_0,$$

$$\frac{dy}{dx}(x_0) = y_1,$$

$$\vdots$$

$$\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$$

where $x_0 \in I$ and $y_0, y_1, ..., y_{n-1}$ are constant. This is called initial value problem.

Example

Show that $\phi(x) = \sin(x) - \cos(x)$ is a solution to the initial value problem

$$\frac{d^2y}{dx^2} + y = 0; \;\; y(0) = -1, \;\; \frac{dy}{dx}(0) = 1.$$

Example

The function $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is a solution to

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

where c_1 and c_2 are constants. Determine c_1 and c_2 so that the initial conditions

$$y(0) = 2 \text{ and } \frac{dy}{dx}(0) = -3$$

are satisfied.

Existence and Uniqueness of Solution

Theorem

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0.$$

if f *and* $\frac{\partial f}{\partial y}$ *are continuous functions in some rectangle*

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0)*, then the initial value problem has a unique solution* $\phi(x)$ *in some interval* $x_0 - \delta < x < x_0 + \delta$*, where* δ *is a positive number.*

Example

For the initial value problem

$$3\frac{dy}{dx} = x^2 - xy^3, \; y(1) = 6,$$

does Theorem imply the existence of a unique solution?

Example

For the initial value problem

$$\frac{dy}{dx} = 3y^{2/3}, \quad y(2) = 0,$$

does Theorem imply the existence of a unique solution?

- Do problems 1,3,7,9,13,15,19, 21, 25, 27,29, and 31, pages 13-15.

- $$\frac{dy}{dx} = f(x, y)$$
- Specifies slope at each point in the xy-plane.
- $$\frac{dy}{dx} = x^2 - y.$$
- The solution that posses at point $(-2, 1)$ must have slope ....

- A direction field is a plot of short line segments at various points in the xy-plane showing the the slope of the solution curve.

Scilab

- Plot the direction vector field of
$$\frac{dy}{dx} = x^2 - y.$$
using scilab
- Plot the direction vector field of
$$\frac{dy}{dx} = -2y.$$
using scilab
- Plot the direction vector field of
$$\frac{dy}{dx} = -y/x.$$
using scilab
- Plot the direction vector field of
$$\frac{dy}{dx} = 3y^{2/3}.$$
using scilab

Scilab

$$\frac{dp}{dt} = p(1 - p).$$

- If the initial population is 3000, what can you say about the limiting population $\lim_{t \to +\infty} p(t)$?
- Can a population 1000 ever decline to 500?
- Can a population of 1000 ever increase to 3000?

```
clear all;
clc
exec diff.sci
p0=.5; t0=0; T=0 : 0.001 : 2;
sol = ode( p0 , t0 , T , diff ) ;
xset('background', 1); // Black background
xset('foreground', 8); // White axes and labels
//xset('font', 3, 10); // Italics and bigger font size
xset('thickness',2);// Thicker lines
plot2d (T, sol,8)
xlabel ('Time t');
```

```
function pdot = diff(t,p)
  pdot =p*(1-p);
  endfunction
```

First order differential equation

- $y' = f(x, y), y(x_0) = y_0$.
- step size: h
- equally space points: $x_n = x_0 + n * h, n = 0, 1, 2, ..$
- initial point: (x_0, y_0)
- iteration along x-axis: $x_1 = x_0 + h$
- iteration along y-axis: $y_1 = y_0 + hf(x_0, y_0)$
- $(x_{n+1}, y_{n+1}) : x_{n+1} = x_n + h, y_{n+1} = y_n + hf(x_n, y_n)$

Example computing by hand

- $y' = x\sqrt{y}, y(1) = 4.$
- step size: $h = 0.1$
- equally space points: $x_1 = 1 + n * 0.1, n = 1, 2, 3, 4, 5.$
- initial point: $(1, 4)$
- iteration along x-axis: $x_1 = x_0 + h = 1 + 0.1 = 1.1$
- iteration along y-axis: $y_1 = y_0 + hf(x_0, y_0) = y_0 + hx_0\sqrt{y_0} = 4 + 0.1(1)(\sqrt{4}) = 4.2$
- $x_2 = x_1 + h = 1.1 + .1 = 1.2$
- $y_2 = y_1 + hf(x_1, y_1) = y_1 + hx_1\sqrt{y_1} = 4.2 + .1(1.1)(\sqrt{4.2}) \approx 4.42453$

```
function y = Euler(y0,x0,x,f )
n = length(t) , y = y0 ;
for j = 1 : n-1
  h=x(j+1)-x(j);
y0=y0+h*f(x(j),y0);
 y=[y y0];
end;
endfunction


clear all;
clc
exec diff2.sci
exec Euler.sci
y0=4; x0=0; X=0 : 0.01 : 2;
sol =Euler( y0, x0 ,X, diff2) ;
xset('background', 1); // Black background
xset('foreground', 8); // White axes and labels
//xset('font', 3, 10); // Italics and bigger font size
xset('thickness',2);// Thicker lines
```

```
plot2d (X, sol ,2) // Blue line
xlabel ('Time t');
```

```
function ydot=diff2(x,y)
 ydot=x\sqrt{y};
 endfunction
```

- $y' = y, y(0) = 1.$
- number of steps:$N = 1, 2, 4, 8$ and 16. $h = 1/N$
- $N = 1$ $h = 1$
- $x_1 = x_0 + h = 0 + 1 = 1$,$y_1 = y_0 + hf(x_0, y_0) = 1 + 1(1) = 2$
- $N = 2$ $h = 1/N = .5$
- $x_1 = x_0+h = 0+.5 = 0.5$,$y_1 = y_0+hf(x_0, y_0) = 1+.5(1) = 1.5$
- $x_2 = x_1 + h = 1$,$y_2 = y_1 + hf(x_1, y_1) = 1.5 + .5(1.5) = 2.25$

Homework

- Do problems 1,3, 5,9,and 13, page 28.

Heun1.sci

```
  // Heun1.sci file for improved Euler Method
  function p = Heun1(p0,t0,t,g )
n = length(t) , p = p0 ;
for j = 1 : n-1
  k1=g(t(j),p0);
  k2=g(t(j+1),p0+(t(j+1)-t(j))*k1);
p0=p0+((t(j+1)-t(j))/2)*(k1+k2);
 p=[p p0];
end;
endfunction
```

logistic.sce

```
    clear all;
clc
exec logistic1.sci
exec Heun1.sci
p0=1; t0=0; T=0 : 0.01 : 2;
sol =Heun1( p0, t0 ,T, diff1) ;
xset('background', 1); // Black background
xset('foreground', 8); // White axes and labels
xset('thickness',2);// Thicker lines
plot2d (T, sol ,2) // Blue line
xlabel ('Time t');
```

logistic1.sci

```
//logistic1.sci
//differential equation file
function pdot = logistic1(t,p)
  pdot =p*(p-2);
  endfunction
```

Rungekutta1.sci

```
  function p = Rungekutta1(p0,t0,t,g )
n = length(t) , p = p0 ;
for j = 1 : n-1
  h=t(j+1)-t(j);
  k1=g(t(j),p0);
  k2=g(t(j)+h/2,p0+(h*k1)/2);
  k3=g(t(j)+h/2,p0+(h*k2)/2);
  k4=g(t(j+1),p0+h*k3);
p0=p0+(h/6)*(k1+2*k2+2*k3+k4);
 p=[p p0];
end;
endfunction
```

system.sce

```
  clear all;
clc
exec spring11.sci
exec Euler1.sci
x0 =[ 1 , 0 ]' ; t0=0; T=0 : 0.1 : 100; T=T';
sol = Euler1( x0 , t0 ,T, pred ) ;
//sol=ode(x0, t0, T, pred );
plot2d (T, sol' )
xset( 'window', 1)
plot2d(sol(2,:),sol(1,:))
```

Euler1.sci

```
  function p = Euler1(p0,t0,t,g )
n = length(t) , p = p0 ;
for j = 1 : n-1
  h=t(j+1)-t(j);
p0=p0+h*g(t(j),p0);
 p=[p p0];
end;
endfunction
```

spring11.sce

```
function xdot=spring11(t,x)
a=1;b=0.1;
xdot(1)=x(2);
xdot(2)=-b*x(2)-a*x(1);
endfunction
```