Problem Set #4

ECON 7010, Prof. Jason DeBacker
Due Tuesday, October 20, 2:40 p.m.

NOTE: Feel free to work in groups on these problems. However, I would like each of you to turn in your own solutions.

1. Consider the optimization problem of a firm which produces output from capital and labor. Assume that the firm’s stock on capital is given and that labor is hired at a real wage w. Suppose the firm is a monopolist in the product market and faces a demand curve with constant elasticity: \( q^d = p^{-\gamma} \). Let \( \Pi(K) \) be the firm’s profit after it has optimally chosen labor given the demand curve. Solve for \( \Pi(K) \).

**ANSWER:** \( \Pi(K) = \max_n q^* p q^\frac{1}{1-\gamma} - n w = \max_n q^* q^\frac{1}{1-\gamma} - n w = \max_n f(K,n)1^{\frac{1}{1-\gamma}} - nw \). The FOC of this problem implies that: \( w = (1 - \frac{1}{\gamma})(f(K,n))^{\frac{\gamma}{1-\gamma}} f_n(K,n) \). If you assume a particular functional form for \( f(K,n) \), you can solve for \( n \) as a function of \( K \), \( w \), and model parameters. E.g., if \( f(K,n) = K^\alpha n^{1-\alpha} \) then you can solve for \( n \) and will find \( n = \left( \frac{w}{(1-\frac{1}{\gamma})(1-\alpha)} \right)^{\frac{1}{\gamma}} K^{\alpha(1-\gamma)} \). Call this function \( n(K) \). Thus \( \Pi(K) = f(K,n(K))^{1^{\frac{1}{1-\gamma}} - nw} \).

2. Consider the optimization problem of the firm with quadratic adjustment costs. Suppose that profit as a function of capital, \( K \), and a shock, \( A \), is given by: \( \Pi(A,K) = AK^\alpha \) with \( \alpha < 1 \). What is the dynamic programming problem for the firm? Show that the value function is NOT proportional to \( K \).

**ANSWER:** The functional equation is: \( V(A,K) = \max_{K'} AK^\alpha - p(K' - (1-\delta)K) - \frac{2}{\gamma} \left( \frac{K' - (1-\delta)K}{K} \right)^2 K + \beta E_{A'|A} V(A',K') \). You then want to show that \( V(A,K) \neq \eta K \) where \( \eta \) is some constant. The best way to do this is to look at the FOC assuming that \( V(A,K) = \eta K \) and then showing that this is inconsistent with the functional equation. The FOC implies that: \( \frac{1}{K} = \frac{1}{\gamma} \left( \beta E_{A'|A} V(A',K') - p \right) \). Assuming that \( V(A,K) = \eta K \), then we know that \( V_{K'}(A',K') = \eta \). Thus the FOC implies that \( K = \frac{1}{\gamma} \left( \beta \eta - p \right) \). You can then solve for \( K' \) from this FOC and you get: \( K' = \left( \frac{1}{\gamma} (\beta \eta - p) + (1-\delta) \right) K \). Plugging this equation for \( K' \) into the FE (and assuming \( V(A,K) = \eta K \), we get: \( \eta K = AK^\alpha - p \left( \frac{1}{\gamma} (\beta \eta - p) \right) K - \frac{2}{\gamma} \left( \frac{1}{\gamma} (\beta \eta - p) \right)^2 K + \beta E_{A'|A} \eta \left( \frac{1}{\gamma} (\beta \eta - p) + (1-\delta) \right) K \). Doing some algebra to solve for \( \eta \), you will find that \( \eta \) is some function of \( K \). Therefore, \( \eta \) is not a constant for all \( K \) and thus the value of the firm is not proportional to the capital stock.

Now consider the non-stochastic growth model presented in class as a starting point. Each problem below outlines a stochastic version of the growth model. For each, write down the appropriate functional equation. Identify the state and control variables. Write down the conditions for optimality. Intuitively, how do the key macroeconomic variables (output, consumption, investment, employment) respond to these shocks?

3. Suppose the production function is stochastic: \( y = AF(k) \). Assume \( A \) follows a first-order Markov process.
4. Suppose that household preferences are given by $u(c, n, \varepsilon)$ where $c$ is consumption, $n$ is labor supply, and $\varepsilon$ is a shock to preferences. Assume that $\varepsilon$ follows a first-order Markov process. What assumptions are you making about $u(\cdot)$? Also, modify the technology so that there is a labor input (recall the class notes on the non-stochastic growth model with labor).

**ANSWER:** I will assume that $u_c(\cdot) > 0$, $u_{cc}(\cdot) < 0$, $u_n(\cdot) < 0$, and $u_{nn}(\cdot) > 0$. I will also assume that the taste shock for the current period is known when making decisions in the current period. With labor, the production technology is: $y = f(n, k)$. The functional equation is: $V(k, n) = \max_{n,k} u(f(n, k)) + (1 - \delta)k - k' + \beta E \max_{k'} V(k', n')$. State variables: $k, \varepsilon, n$. The FOCs imply the (inter temporal) Euler equation, $\varepsilon u_c(c, n) = \beta E \varepsilon u_c(c', n')(f_k(n', k') + (1 - \delta))$ and the (intratemporal) condition from the FOC w.r.t. labor supply: $f_n(n, k) u_c(c, n) = -u_n(c, n)$. Since $\varepsilon$ is a shock to utility, the planner will increase consumption and decrease investment when the shock is higher. Further, because of the disutility of labor, employment will decrease when the shock is higher (note that even though the $\varepsilon$ cancels out of the FOC w.r.t. labor supply, because of the different $c$ when the shock is higher, this implies a different $n$ as well). Because of the decline in labor supply, output will also decline when the shock is higher.

5. Return to the original problem where utility is just $u(c)$ and technology is $y = f(k)$. Suppose that the accumulation technology is given by $k' = k(1 - \delta) + \varepsilon$, where $\varepsilon$ follows a first-order Markov process. What do you think the correlation will be between consumption and investment in this economy?

**ANSWER:** Assume that know $\varepsilon$ at the time of the choice of $k'$. Functional equation: $V(k, \varepsilon) = \max_i u(f(k) - i) + \beta E V(k', \varepsilon')$. States: $k, \varepsilon$, controls: $i$ (or, equivalently, $k'$). Euler: $u'(c) = \beta E u'(c') f'(k')$. Here the shock increases/decreases net investment in a way that is proportional to the size of the shock. An increase in $\varepsilon$ will increase next period capital stock and thus consumption. Therefore, consumption smoothing implies that an indirect effect of the shock is to increase consumption today and decrease investment. At the same time, the direct effect of the investment shock increases investment (since the planner now gets more capital next period for each unit of investment this period with the higher shock). These effects are larger if the shock has more persistence. Future output increases with a higher shock today. There is not labor in this model, so employment is not affected.