Question 1
Consider the two-period cake eating problem. The agent is endowed with a cake of size $w_1$ at beginning of period 1 and receives income in the form of cake of amount $y_2$ at the beginning of period 2. The agent derives utility from the consumption of cake in periods 1 and 2, discounts period 2 at rate $\beta$, and can borrow and lend freely. Assume that $u'(c) > 0$, $u''(c) < 0$, and $u'(0) = \infty$.

(a) What is (are) the state variable(s)? The control variable(s)? (5)
ANSWER: states = $w_1, w_2$. Controls = $w_2, s, c_1, c_2$

(b) What is (are) the transition equation(s)? (5)
ANSWER: $w_2 = w_1 - c_1 + y_2$ or $w_2 = s + y_2$

(c) Formulate the dynamic programming problem of the cake eater. (15)
ANSWER: The Bellman equation for the cake eater’s problem is: $V_2(w_1) = \max_{c_1, c_2} u(c_1) + \beta u(c_2)$ or $V_2(w_1) = \max_s u(w_1 - s) + \beta u(s + y_2)$

(d) What is (are) the necessary condition(s) for optimization. Interpret. (20)
ANSWER: the FOC is given by the Euler equation: $u'(w_1 - s) = \beta u'(s + y_2)$. This equation says that, at an optimum, the agent chooses the levels of consumption (equivalently, savings) such that the discounted marginal utilities of consumption are equated across periods.

(e) Suppose that $u(c) = \ln(c)$, solve for the policy functions for saving and consumption explicitly. (20)
ANSWER: In this case, the Euler equation is given by:

$$\frac{1}{w_1 - s} = \frac{\beta}{s + y_2}$$

$$\implies \beta w_1 - \beta s = s + y_2$$

$$\implies (1 + \beta)s = \beta w_1 - y_2$$

$$\implies s = \frac{\beta w_1 - y_2}{1 + \beta}$$

With the solution to the savings function, we can then use the transition equations to find: $c_1 = \frac{w_1 + y_2}{1 + \beta}$ and $c_2 = \beta \frac{w_1 + y_2}{1 + \beta}$.

(f) Use comparative statics to show that savings is increasing in the amount of the endowment ($w_1$) and decreasing the amount of income ($y_2$) given the parameterization in part (e). (20)
ANSWER: Since we have the explicit solution to the policy functions, we do not need to use the implicit function theorem. We can simply take the derivative of the policy functions. Here we find $\frac{\partial s}{\partial w_1} = \frac{\beta}{1 + \beta}$ and $\frac{\partial s}{\partial y_2} = -\frac{1}{1 + \beta}$.
(g) Formulate the dynamic programming problem of the cake eater in the case where she has uncertainty about her income in period 2, $y_2$. 

\text{ANSWER: } V_2(w_1) = \max_s u(w_1 - s) + E_{y_2} u(s + y_2)

\section*{Question 2}

Consider the infinite-horizon, deterministic cake eating problem presented in class.

(a) Show whether or not $w' = \beta w$ is the policy function if $u(c) = \ln(c)$. 

\text{ANSWER: The Euler equation is given by: } u'(w - w') = \beta u'(w' - w''). 

Given the log utility, we have:

\[ \frac{1}{c} = \beta c' \]

\[ \Rightarrow \frac{1}{w - w'} = \frac{\beta}{w' - w''} \]

plugging in the guess gives:

\[ \Rightarrow \frac{1}{w - \beta w} = \frac{\beta \beta w - \beta w'}{\beta \beta w - \beta w''} \]

\[ \Rightarrow \frac{1}{w - \beta w} = \beta \beta w - \beta w' \]

\[ \Rightarrow \frac{1}{w - \beta w} = \beta \beta w - \beta w \]

\[ \Rightarrow \frac{1}{w - \beta w} = \beta \beta w - \beta w \]

This last line is true for all $w$. Therefore, we have shown that this guess was in fact the solution to the policy function.

(b) What is the value function, $V(w)$? Can you solve for it with the information given? 

\text{ANSWER: Since we know the correct policy function, we can just put this into the Bellman equation: } V(w) = \max_{w'} u(w - w') + \beta V(w'). 

Doing so yields: $V(w) = \ln(w - \beta w) + \beta V(\beta w)$. 

But we are stuck here - we can’t solve for $V(w)$ explicitly. However, we could write out the infinite series:

\[ V(w) = \ln(w - \beta w) + \beta \ln(\beta w - \beta^2 w) + \beta^2 \ln(\beta^2 w - \beta^3 w) + ... \]

\[ \Rightarrow V(w) = \ln(1 - \beta) + \ln(w) + \beta [\ln(\beta) + \ln(1 - \beta) + \ln(w)] + \beta^2 [\ln(\beta^2) + \ln(1 - \beta) + \ln(w)] + ... \]

\[ \Rightarrow V(w) = \sum_{t=0}^{\infty} \beta^t \ln(w) + \sum_{t=0}^{\infty} \beta^t \ln(1 - \beta) + \sum_{t=0}^{\infty} \beta^t \ln(\beta^t) \]

\[ \Rightarrow V(w) = \frac{1}{1 - \beta} \ln(w) + \frac{1}{1 - \beta} \ln(1 - \beta) + \sum_{t=0}^{\infty} \beta^t \ln(\beta^t) \]

(3)

So in this way you can solve for $V(w)$ explicitly. The answer you need to have doesn’t require this, you just need to show some thought.
Question 3
Mitt Romney has been watching the Republican debates in despair. He’s considering another run for president. He’s trying to make the discrete choice - run or not? Each period, Romney’s probability of a victory if he joins the race in that period, $P_i$, fluctuates between $P_l$ and $P_h$, where $P_l < P_h$.

The transition between these probabilities follows a first order Markov process and is summarized by the matrix $\Pi$ (which has as its elements $\pi_{i,j}$, giving the probability of moving from state $i$ to state $j$). If he wins office, Romney will receive a payoff from the rents to holding office of $R$ in each period after he wins (not just for the term of his presidency). He receives a payoff of 0 if he does not win, the same payoff he realizes in periods in which he does not run for office. Romney discounts future periods at rate $\beta$.

(a) What are the states and what are the controls? What is (are) the transition equations? (30)

ANSWER: State variable: $P_i, i \in \{l, h\}$. Control: run, not run. Transition: $\Pi$

(b) Write down Romney’s dynamic programming problem. (30)

ANSWER: $V(P_i) = max \{V^0(P_i), V^1(P_i)\}$. The value of not running is given by $V^0(P_i) = \beta E_{P'} P_i V(P')$ and the value of running is given by $V^1(P_i) = P_i u(R) + (1 - P_i) u(0)$. We’ll assume that $u(0) = 0$ in what follows to simplify this as: $V^1(P_i) = P_i u(R)$, but you need not do it this way.

(c) Assuming that there is some state in which Romney runs for office, give the conditions under which he will enter the race. (40)

ANSWER: If Romney is ever going to enter, he’ll enter when his chances are high for sure. Thus $V(P_h) = P_h u(R)$. He may or may not enter the race when his odds of winning are low. To determine when he’ll enter, use a threshold rule. He’ll enter in the low state if the following inequality holds:

$$P_l u(R) \geq \beta E_{P'} P_l V(P')$$

$$\implies P_l u(R) \geq \beta \left[ \pi_{lh} P_h u(R) + \pi_{ll} \beta E_{P'} P_l V(P') \right]$$

Where the second line follows from the fact that he runs in the high state

$$\implies P_l u(R) \geq \beta \frac{\pi_{lh} P_h u(R)}{1 - \beta \pi_{ll}}$$

$$\implies P_l \geq \beta \frac{\pi_{lh} P_h}{1 - \beta \pi_{ll}}$$

Where the second to last line follows by solving $E_{P'} P_l V(P') = \pi_{lh} P_h u(R) + \pi_{ll} \beta E_{P'} P_l V(P') \implies E_{P'} P_l V(P') = \frac{\pi_{lh} P_h u(R)}{1 - \beta \pi_{ll}}$. This inequality makes sense - Romney is more likely to run in the low state if he has a low $\beta$ or if $\pi_{lh}$ is small. He’s also less likely to run in the low state if the probability of winning in the high state is much greater than that in the low state. Note that the rents from office do not matter in his decision.