Chapter 5 - Choice

→ Now that we've defined what is affordable (budget constraint) and what is preferred, we put them together to see how consumers choose the most preferred bundle from their budget sets.

Optimal Choice

→ To solve the consumer's problem, we'll use calculus.

→ Note that this problem is constrained!

→ Consumption expenditures are constrained by the budget set.

→ Consumption must be non-negative.

→ So the general problem can be stated as:

\[
\max_{x_1, x_2} u(x_1, x_2)
\]

subject to \( p_1 x_1 + p_2 x_2 \leq m, \quad x_1 \geq 0, \quad x_2 \geq 0 \)

\((x_1^*, x_2^*)\) will be the consumption bundle that solves this problem.

→ We've seen how to solve an unconstrained maximization problem before.

→ Find where the slope = 0

→ To solve a constrained problem, we will introduce a new tool - the Lagrangian.
The Lagrangian

- a way to incorporate the objective function (e.g., $ux, x_2$) and the constraints into a single function to be maximized.

- we will incorporate the constraints with Lagrange multipliers.

- there are constants $\lambda$ that penalize the Lagrangian function if the constraint is violated.

- this ensures that the constraints are not violated - you can't maximize the Lagrangian via violation.

Example: Cobb-Douglas utility

$\max \quad x_1 x_2$

$s.t. \quad px_1 + px_2 x_2 \leq m$

$x_1 \geq 0$

$x_2 \geq 0$

The Lagrangian is:

$L = x_1 x_2 + \lambda (m - px_1 - px_2) + m_1 x_1 + m_2 x_2$

Objective function

Lagrange multipliers on BE

Lagrange multipliers on the non-negativity constraints on consumption

→ note how constraints are enforced:

- they are enforced such that if they are violated, they lower the value of $L$.

- e.g., it spends more than income when $m - px_1 - px_2 < 0$. 
These Lagrange multipliers are all non-negative:

\[ \lambda \geq 0 \]
\[ \mu_1 \geq 0 \]
\[ \mu_2 \geq 0 \]

They also have an interpretation that we'll see more clearly in a moment:

these multipliers will represent the value of relaxing the constraint

Lagrange's theorem says that the solution to the constrained maximization problem satisfies the following conditions:

\[ \frac{\partial \lambda}{\partial x_1} = c x_1 x_2 d - \lambda p_1 \Rightarrow \mu_1 = 0 \]
\[ \frac{\partial \lambda}{\partial x_2} = d x_1 x_2 d - \lambda p_2 \Rightarrow \mu_2 = 0 \]

\[ \lambda \frac{\partial \lambda}{\partial \lambda} = 0 \Rightarrow \frac{\partial \lambda}{\partial \lambda} = p_1 x_1 + p_2 x_2 - m = 0 \]
\[ \text{or} \quad \lambda = 0 \]

\[ \mu_1 \frac{\partial \lambda}{\partial \mu_1} = 0 \Rightarrow \frac{\partial \lambda}{\partial \mu_1} = x_1 = 0 \]
\[ \text{or} \quad \mu_1 = 0 \]

\[ \mu_2 \frac{\partial \lambda}{\partial \mu_2} = 0 \Rightarrow \frac{\partial \lambda}{\partial \mu_2} = x_2 = 0 \]
\[ \text{or} \quad \mu_2 = 0 \]
Notice what the necessary conditions are telling us:

- either don't spend all money (hand \( \lambda = 0 \)) or do and \( \lambda > 0 \)
- either consume positive \( x_1 \) (and \( \mu_1 = 0 \)) or don't (and \( \mu_1 > 0 \))
- either consume positive \( x_2 \) (and \( \mu_2 = 0 \)) or don't (and \( \mu_2 > 0 \))

and regarding the first 2 necessary conditions:

- the slope will not equal zero at the maximum if the constraints bind

Solving the constrained optimization problem

- we have 5 equations and 5 unknowns \( (x_1, x_2, \lambda, \mu_1, \mu_2) \)

\[ \begin{align*}
\text{(1)} & \quad cx_1^{\alpha - 1} x_2^{\beta} + \mu_1 = \lambda P_1 \\
\text{(2)} & \quad dx_1^{\alpha - 1} x_2^{\beta - 1} + \mu_2 = \lambda P_2
\end{align*} \]

Note that if \( x_1 = 0 \), then \( \text{(1)} \Rightarrow \mu_1 = \lambda P_1 > 0 \)

\( \Delta \lambda = 0 \)

But then \( \text{(2)} \Rightarrow \mu_2 > 0 \Rightarrow x_2 = 0 \)

But then BC not binding

\( \Rightarrow \lambda = 0 \)

\( \Rightarrow x_1 > 0, x_2 > 0 \)
Blc \( x_1 > 0, x_2 > 0, \mu_1 = \mu_2 = 0 \)

So

1. \( \Rightarrow \) becomes: \( c \cdot x_1^{c-1} x_2^d = \lambda P_1 \)
2. \( \Rightarrow \) becomes: \( d x_1^{c-1} x_2^{d-1} = \lambda P_2 \)

- dividing \( \#1 \) by \( \#2 \) we get:

\[
\frac{c x_1^{c-1} x_2^d}{d x_1^{c-1} x_2^{d-1}} = \frac{P_1}{P_2}
\]

\[c x_2 = \frac{P_1}{P_2} \]

**MRS**

\( = \) **Price Ratio**

What this means is slope of indifference curve equals slope of budget line.

We've seen this:

![Indifference Curves Diagram](image)

- the highest indifference curve is the one that just
- "just touches" means is tangent to the BC
- tangent means have the same slope at that point
So the calculus of our FOCs gives the same solution we got by looking at the graph.

we can continue with our equations to solve for our demand for $x_1$ and $x_2$:

$$\frac{C x_2}{dx_1} = \frac{p_1}{p_2}$$

$$x_2 = \frac{p_1 dx_1}{p_2 c}$$

$$(x_2(x_2)) = \text{plug this into } \frac{C x_2}{dx_1}:$$

$$p_1 x_1 + p_2 x_2 = m$$

$$p_1 x_1 + x_2 \frac{p_1}{p_2 c} x_1 = m$$

$$p_1 x_1 (1 + \frac{dx_1}{c}) = m$$

$$p_1 x_1 \left( \frac{c + d}{c} \right) = m$$

$$x_1 = \frac{m}{p_1} \frac{c}{c+d}$$

$$x_2 = \frac{p_1 d x_1}{p_2 c} = \frac{p_1 m c d}{p_1 p_2 c + c + d}$$

$$x_2 = \frac{m d}{p_2 c + d}$$
\[ x_1 = \frac{m}{p_1} \frac{c}{\text{c+d}} \]

\[ x_2 = \frac{m}{p_2} \frac{d}{\text{c+d}} \]

\[ x_2 = \frac{m}{p_2} \frac{d}{\text{c+d}} \]

\[ \text{to make monotonically transform utility function such that} \]
\[ \text{c+d = 1} \]
\[ \text{let} \quad a = \frac{c}{\text{c+d}} \]
\[ 1-a = \frac{d}{\text{c+d}} \]
\[ \frac{\text{c+d}}{c} = \frac{c}{c+d} \]
\[ x_1, x_2 = \frac{x_1}{\text{c+d}} \]
\[ = x_1 \frac{1}{a} \]
\[ = x_1 x_2 \]

\[ \text{this will rep the same preferences} \]

\[ x_1 = \frac{m}{p_1} \]
\[ x_2 = \frac{m}{p_2} (1-a) \]

\[ \text{what these mean} \]
\[ \text{P1x1 = 1}\text{a} \]
\[ \text{P2x2 = (1-a) m} \]
What about the constraints?

- We solve for \( \mu_1 = \mu_2 = 0 \)
- Non-negativity constraint doesn't bind

- We can solve for \( \lambda \) from

\[
\begin{align*}
\frac{c_{x_1} x_1 - 1}{x_1} = \frac{c_{x_2} x_2}{x_2} = \lambda
\end{align*}
\]

- Not that \( \lambda \) also equals the

- Mill per dollar on \( x_1 \)

- Which makes sense if these

- Amounts weren't the same,

- The consumer wouldn't be

- At an optimum - she could

- Spend a little less on one

- Good and more on the

- Other.
Example: Project Substitutes

\[ u(x_1, x_2) = x_1 + x_2 \]

\[ \Rightarrow \lambda = x_1 + x_2 + \lambda (m - p_1 x_1 - p_2 x_2) + \mu_1 x_1 + \mu_2 x_2 \]

Fix \( x_2 \):

1. \[ \frac{\partial u}{\partial x_1} = 1 - \lambda p_1 + \mu_1 = 0 \]

2. \[ \frac{\partial u}{\partial x_2} = 1 - \lambda p_2 + \mu_2 = 0 \]

3. \[ \frac{\partial u}{\partial x} = p_1 x_1 + p_2 x_2 - m = 0 \quad \Rightarrow \lambda = 0 \]

4. \[ \frac{\partial u}{\partial \mu_1} = x_1 = 0 \quad \Rightarrow \mu_1 = 0 \]

5. \[ \frac{\partial u}{\partial \mu_2} = x_2 = 0 \quad \Rightarrow \mu_2 = 0 \]

1. \[ \Rightarrow 1 + \mu_1 = \lambda p_1 \]

2. \[ \Rightarrow 1 + \mu_2 = \lambda p_2 \]

\[ \Rightarrow \lambda > 0 \Rightarrow BC binds (spend all money) \]

\[ \Rightarrow x_1 > 0 \text{ or } x_2 > 0 \text{ or both} \]

2. \[ \Rightarrow 1 + \mu_2 = \lambda p_2 \]

\[ \Rightarrow \text{nothing eliminates case that one of } \mu \text{ demands} = 0 \]

\[ \Rightarrow \text{this is called a corner sol'n} \]
What we'll do then is to consider an interior sol'n. Then consider the corner sol'ns (all \( x_1 \) or all \( x_2 \)) and see what conditions on prices put us there:

1. \( x_1 \) and \( x_2 > 0 \):

\[ \Rightarrow M_1 = M_2 = 0 \]

\[ \Rightarrow 1 = \lambda p_1 = \lambda p_2 \]

\[ \Rightarrow \lambda = p_2 \]

\[ \Rightarrow \text{no interior sol'n only if } p_1 = p_2 \]

\[ \Rightarrow \lambda = \frac{p_1}{p_2} \]

\[ \Rightarrow x_2 \leftarrow \frac{1}{\lambda} IC \rightarrow \text{ slope } = -1 \]

when the line tells right

\[ \Rightarrow MBL, \, \text{ then have interior sol'n} \rightarrow \text{ consumer indifferent between any point on BL} \]

2. \( x_1 > 0, \, x_2 > 0 \):

\[ \Rightarrow M_1 = M_2 = 0 \]

\[ \Rightarrow 1 = \lambda p_1 \]

\[ \Rightarrow \lambda p_2 = \lambda p_2 \]

\[ \Rightarrow \lambda p_1 > \lambda p_2 \]

\[ \Rightarrow p_1 > p_2 \]
If \( p_1 < P_2 \), consume only \( x_1 \):

- If \( x_2 > 0, x_1 = 0 \)
  - \( M_1 > 0, M_2 = 0 \)
  - \( 1 - M_1 = \lambda P_1 \)
  - \( 1 = \lambda P_2 \)
  - \( \lambda P_1 > \lambda P_2 \)
  - \( P_1 > P_2 \)
  - If \( P_1 > P_2 \), consume only \( x_2 \)

\[ x_2 \]

\[ x_1 \]
-> consider other cases
  -> Rods
  -> Nut ands
  -> Concave prof

-> The Lagrangian with all constraints will always work
  -> but can be harder to remember
  where corner so likely

-> also, care like perfect complements
don't have derivative:

\[ u(x_1, x_2) = \min \{ax_1, bx_2^3 \} \]

-> so think graphically: --

\[ x_2 \]
\[ x_1 \]
utility functions and optimal taxes

Consider 2 taxes:

1. A quantity tax on good 1 at a rate of \( t \)
2. A lump sum tax, \( T \)

Let's make the sizes of the taxes the same - so that the revenue raised from the quantity tax at the consumer's optimal choice, \( x^* \), is the same as that raised from the lump sum tax, \( T \):

\[ T = tx^* \]

With a quantity tax:

Maximize

\[ u(x_1, x_2) \]

Subject to \( (p_1 + t)x_1 + px_2 \leq w \)

Assume \( w = w' \)

\[ x_1 = x_1(x_2) + \lambda (w - (p_1 + t)x_1 - px_2) \]

\[ \frac{\partial x_1}{\partial x_2} = \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda (p_1 + t) = 0 \]

\[ \frac{\partial x_2}{\partial x_2} = \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \]

\[ \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{p_1 + t}{p_2} \]

\[ \frac{\partial u(x_1, x_2)}{\partial x_2} \]

MRS: slope of BC
\[ \text{Lump sum tax:} \]

\[ \begin{align*}
\text{max} & \quad u(x_1, x_2) \\
\text{s.t.} & \quad p_1 x_1 + p_2 x_2 \leq M - T \\
& \quad \text{consume interior sol'n}
\end{align*} \]

\[ \Rightarrow \quad \lambda = u(x_1, x_2) + \lambda (M - T - p_1 x_1 - p_2 x_2) \]

\[ \frac{\partial \lambda}{\partial x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0 \]

\[ \frac{\partial \lambda}{\partial x_2} = \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \]

\[ \frac{\partial u(x_1, x_2)}{\partial x_1} \bigg| \frac{\partial u(x_1, x_2)}{\partial x_2} = \frac{p_1}{p_2} \]

\[ \text{MRS} \quad \text{slope of BL} \]

\[ \Rightarrow \quad \text{since } x_1^*(T) \text{ affordable under lump sum tax} \]

\[ \Rightarrow \text{tax } \Rightarrow \text{ then lump sum tax makes consumer at least as well off} \]