Chap 4 - Utility

→ We'll use the term "utility" to refer to an economic consumer's well-being.

→ To put this in the context of consumer preferences, from Chap 3, we'll say that bundle \((x_1, x_2)\) gives the consumer more utility than \((y_1, y_2)\) if:

\[(x_1, x_2) \succ (y_1, y_2)\]

→ A utility function is a function that assigns a value to every possible consumption bundle.

Types of Utility Functions

→ We can split utility functions into 2 types: ordinal and cardinal.

→ Ordinal utility functions assign values to consumption bundles in a way that preserves the consumer's preference ordering, but does not give information about the "magnitude" of those preferences.

→ E.g.: Alex prefers the following preferences: Apple > Bananas > Chocolates, we could state the following ordinal utility function:

<table>
<thead>
<tr>
<th>Bundle</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
</tbody>
</table>

→ This preserves the order
→ It doesn't make sense to say that bundle A gives 25% more utility than B or C.

Observe another utility function that also appears consistent with Alex's preferences.
Bundle | Utility
--- | ---
A | 25
B | 16
C | 9

→ The utility function also represents A > B > C

More generally, ordered utility functions have the property that they represent the same preferences under any monotonic transformation.

![Monotonic Transformation](image)

→ A monotonic transformation of a utility function (where the function is non-decreasing) can be given by applying the function f(u) to u.

\[ f(u) = u \times 2 \]

Bundle | Utility
--- | ---
A | 50
B | 10
C | 36

→ Same preference ordering = represent same ordered utility function
Examples of utility functions

First, we'll start with a utility function and show the indifference curves it yields.

Next, we look at indifference curves and find the utility function that gets them.

Recall that an indifference curve is a set of consumption bundles the consumer is indifferent between.

Thus, the utility from any consumption bundle on the indifference curve will be the same.

(i.e., the set of \((x_1, x_2)\) such that \(u(x_1, x_2)\) equals a constant is an indifference curve.

Thus is the set of \((x_1, x_2)\) that result in \(u(x_1, x_2)\) equal to another constant represent another indifference curve.

As an example, consider \(u(x_1, x_2) = x_1 x_2\).

To solve for the an indifference curve, set the utility from \((x_1, x_2)\) equal to some constant, \(k\):

\[u(x_1, x_2) = x_1 x_2 = k\]

\[\Rightarrow x_2 = \frac{k}{x_1}\]

This describes the indifference curve.
Graphically:

- Now consider \( v(x_1, x_2) = x_1 x_2^2 \)
  
  \[ v(x_1, x_2) = x_1 x_2^2 = (x_1 x_2)^2 = u(x_1, x_2) \]

  \( \Rightarrow v(x_1, x_2) \) a monotonic transformation of \( u(x_1, x_2) \)

  \( \Rightarrow \) two will represent the same pays and have the same indifference curves

To see:

Indiff. curve given by \( v(x_1, x_2) = c \), where \( c \)

in some constant

\( \Rightarrow v(x_1, x_2) = x_1 x_2^2 = c \)

\( \Rightarrow x_2 = \frac{c}{x_1 x_2^2} \)

\( \Rightarrow x_2 = \frac{\sqrt{c}}{x_1} \)

So \( \sqrt{c} = k \) (or \( c = k^2 \)) then same - and ok to do this

be utility only ordind - value
Now let's consider some preferences where we can identify curves we know and see if we can recover the utility function that yields them.

**Perfect Substitutes:**

\[ u(x_1, x_2) = x_1 + x_2 \]

- does \( u(x_1, x_2) = x_1 + x_2 \) represent this?
- indiff curve given by \( x_1 + x_2 = k \)
- \( x_2 = k - x_1 \)
  - intercept \( k \)
  - slope \( -1 \)

→ Also, as \( x_1 \) or \( x_2 \) ↑ move to higher indiff curve, which are more preferred bundles

→ So \( u(x_1, x_2) = x_1 + x_2 \) represents these preferences

→ In general, perfect subs are represented by:

\[ u(x_1, x_2) = ax_1 + bx_2 \]

where \( a \) and \( b \)

measure the "value" of goods 1 and 2 to the consumer.

**Indiff curve given by**

\[ ax_1 + bx_2 = k \]

\[ x_2 = \frac{k - ax_1}{b} \]

- intercept \( \frac{k}{b} \)
- slope \( -\frac{a}{b} \)

→ Of course any monotonic transform of this would work.
Perfect Complements

Recall indifference curves "L-shaped":

Consider case of shoes: what matters for utility is the number of complete pairs.

This is given by the minimum number of right or left shoes.

Here $U(x_1, x_2) = \min(x_1, x_2)$ would work (or any monotonically transfrom)

In general, for perfect complements,

$U(x_1, x_2) = \min(a x_1, b x_2)$ describes these preferences, where $a$ and $b$ indicate the proportions in which the goods are consumed.

For example:

$a = \frac{1}{4}$ tbs, $x_1$ = jelly

$b = \frac{1}{2}$ tbs, $x_2$ = peanut butter

Careful: $x_1$ and $b$

2 tbs PB to 1 tbs jelly means $b = \frac{1}{2} = \frac{2 \text{ tbs}}{4 \text{ tbs PB}}$
Quasi-linear Approaches

- These phones are represented by indigo curves that are shifted vertically.

\[ X_2 = k - v(x_1) \text{ where } \]

\( k \) is a constant.

- So again for IC we use: \( X_2 = k - v(x_1) \) where

\[ u(x_1, x_2) = v(x_1) + x_2 \]

- Linear in \( x_2 \)

- Hence name "quasi-linear" (i.e. partly linear).

- Usually better to work with.
one very common utility function is the Cobb-Douglas utility function.

This function is of the form:
\[ u(x_1, x_2) = x_1^c x_2^d \]

where \( c, d > 0 \)

c, d describe curvature of utility curves.

A nice property of this utility function is that it will ensure at least some of each good is consumed.

Just like in any other utility function, a monotonic transformation of the Cobb-Douglas function will represent the same preferences.

For example:
\[ v(x_1, x_2) = u(x_1, x_2)^{1/\gamma} = \left( x_1^c x_2^d \right)^{1/\gamma} = x_1^{c/\gamma} x_2^{d/\gamma} \]

Now define \( a = \frac{c}{cd} \)
We have
\[ V(x_1, x_2) = x_1^{a_1} x_2^{1-a_1} \]

\[ a_1 + a_2 = 1 \]

This means we can always take a monotonically transformed Cobb-Douglas function where the exponents sum to one.

We'll see later on that this will have a useful interpretation.

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Marginal Utility

Marginal utility will define the rate of change in the utility function w.r.t. a change in the consumption of a good.

Marginal just means derivative.

This will be a key concept.

Formally, we define the marginal utility of good 1 as:

\[
MU_1 = \lim_{\Delta x_1 \to 0} \frac{U(x_1 + \Delta x_1, x_2) - U(x_1, x_2)}{\Delta x_1} = \frac{\partial U(x_1, x_2)}{\partial x_1}
\]

Partial derivative of \( U(x_1, x_2) \) w.r.t. \( x_1 \)

Analogous for \( MU_2 \)
Example: \( u(x_1, x_2) = x_1^{\frac{1}{2}} x_2 \)

\[
MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}
\]
\[
= \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}
\]
\[
= \frac{1}{2} \left( \frac{x_2}{x_1} \right)^{\frac{1}{2}}
\]

\[
MU_2 = \frac{\partial u(x_1, x_2)}{\partial x_2} = \frac{1}{2} x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}}
\]
\[
= \frac{1}{2} x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}}
\]
\[
= \frac{1}{2} \left( \frac{x_1}{x_2} \right)^{\frac{1}{2}}
\]

Marginal Utility and the MRS

- Recall our definition of the MRS:
  - MRS = slope of indifference curve at a given point of consumption
  - Told us how much of one good we'd need to give up to get another
  - So, the slope of the IC is negative
  - \( MRS = \frac{\Delta x_2}{\Delta x_1} \)
  - Note that the MRS is the slope of the IC tells us that at a derivative will be involved.

- 2 ways to solve for the MRS:
Using the total differentiated of the utility function:

\[ du = 0 \]

Since utility doesn't change, which is true along the indifference curve.

\[ du = \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0 \]

\[ \Rightarrow \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 = -\frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 \]

\[ \Rightarrow \frac{\partial u(x_1, x_2)}{\partial x_1} \frac{dx_1}{dx_2} = -\frac{\partial u(x_1, x_2)}{\partial x_2} \frac{dx_2}{dx_1} \]

\[ \Rightarrow \text{MRS} = \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{dx_2}{dx_1} \]
MRS = the negative of the ratio of marginal utilities.

2. Using implicit functions:

An indifference curve is defined by:

\[ u(x_1, x_2, x_1(x)) = k \]

Where \( x_2(x_1) \) gives the value of \( x_2 \) that ensures the utility is equal to \( k \) given \( x_1 \).

To find the derivative of both sides w.r.t. \( x_1 \):

\[ \frac{du(x_1, x_2, x_1(x))}{dx_1} + \frac{du(x_1, x_2(x_1))}{dx_2} \frac{dx_2(x_1)}{dx_1} = \frac{dk}{dx_1} \]

By chain rule:

\[ \frac{du(x_1, x_2(x_1))}{dx_1} + \frac{du(x_1, x_2(x_1))}{dx_2} \frac{dx_2(x_1)}{dx_1} = 0 \]

Solve for \( x_2(x_1) \) in the MRS:

\[ \frac{dx_2(x_1)}{dx_1} = -\frac{\frac{du(x_1, x_2(x_1))}{dx_1}}{\frac{du(x_1, x_2(x_1))}{dx_2}} \]

\[ \frac{dx_2(x_1)}{dx_1} = -\frac{2u(x_1, x_2(x_1))}{\frac{du(x_1, x_2(x_1))}{dx_1}} \]

\[ \frac{dx_2(x_1)}{dx_1} = -\frac{2u(x_1, x_2(x_1))}{\frac{du(x_1, x_2(x_1))}{dx_2}} \]

\[ \text{ratio of MRS} \]
So the MRS is the negative of the ratio of marginal utilities.

i.e. \( \text{MRS} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} \)

Example: Cobb-Douglas Utility

\( u(x_1, x_2) = x_1^c x_2^d \)

\( \text{MRS} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} \)

\( \frac{\partial u}{\partial x_1} = cx_1^{c-1} x_2^d \)

\( \frac{\partial u}{\partial x_2} = d x_1^c x_2^{d-1} \)

\( \Rightarrow \text{MRS} = -\left( \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} \right) \)

\( = -\left( \frac{cx_1^{c-1} x_2^d}{dx_1^c x_2^{d-1}} \right) = -\left( \frac{cx_2}{dx_1} \right) \)
Once we know the mathematical relationships, we can easily show that preferences, defined by $U$, remain unchanged under monotonic transformations of the utility function:

$$U(x_1, x_2) = \frac{\partial U(x_1, x_2)}{\partial x_1} = x_1 x_2$$

$$MRS = \frac{\partial U(x_1, x_2)}{\partial x_2} = \frac{x_1}{x_2}$$

$$MRS = -\frac{\partial U(x_1, x_2)}{\partial x_1} = -\frac{x_2}{x_1}$$

$$\text{same as } MRS \text{ for } U(x_1, x_2)$$

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**Applying this**

Varian gives a nice example of how economists have applied utility functions

**Assume utility from transportation given as**

$$U(TW, TT, C) = \beta_1 TW + \beta_2 TT + \beta_3 C$$

where $TW = \text{total walking time (minutes)}$

$TT = \text{total travel time (minutes)}$

$C = \text{trip cost ($\$$)}$

**Econometric techniques** can estimate this function to determine people's choices of transportation methods.
→ the value of $\frac{u(CYW|TT, C)}{u(C)}$ is not important, but the MRS is important and informative

$$MRS_{YW|TT, C} = -\frac{\frac{\partial u}{\partial Y}}{\frac{\partial u}{\partial W}} = -\frac{\beta_2}{\beta_3}$$

Demand and McFadden indices

$\beta_1 = -0.147$
$\beta_2 = -0.0411$
$\beta_3 = -2.24$

$\Rightarrow MRS_{YW, C} = \frac{-0.0411}{-2.24} = 0.0183$

→ willing to pay 0.0183 dollars per minute travel time reduced

→ this is informative for transport policy (among other things)