Mathematics Review

Functions:
- Functions of one argument: $y = f(x)$
  - e.g., $y = x^2$
- Functions of two arguments: $y = f(x_1, x_2)$
  - e.g., $y = x_1 + x_2$
- Functions of many arguments: $y = f(x_1, x_2, ..., x_N)$
  - e.g., $y = \sum_{i=1}^{N} x_i = x_1 + x_2 + ... + x_N$
  - The symbol $\sum$ denotes a summation.

Graphing functions:
- We’ll often understand the economics functions we work with graphically.
- We’ll usually look at functions only in two dimensions since it’s easier to represent and thus visualize on your two dimensional sheet of paper.
- Typically, we’ll put the argument of the function (e.g., $x$) on the horizontal axis and the value of the function (e.g. $y$) on the vertical axis.
- e.g., $y = 2x$:

\[ y = 2x \]

- e.g., $y = x^2$:
Properties of functions:

- A function is **continuous** if it has no kinks or corners.
  - One can take the derivative of a continuous function at any point.
  - We say that a function is $C^1$ if its first derivative exists.
  - We say that a function is $C^2$ if its first and second derivatives exist and so on (e.g., $C^3$, ..., $C^k$)
  - e.g., $y = x^2$ is continuous
    * The first derivative, $\frac{dy}{dx} = 2x$ is also continuous
    * The second derivative, $\frac{d^2y}{dx^2} = 2$, is also continuous.

- A function is **monotonic** if it always increases or always decreases.
  - e.g., $y = 2x$ is “positive monotonic” (or “monotonically increasing”) since $y$ always increases as $x$ increases.
  - e.g., $y = \frac{1}{x}$ is “negative monotonic” (or “monotonically decreasing”) since $y$ always decreases as $x$ increases.

Common functions and their properties:

- **Linear Functions**
  - Linear functions have the form: $y = ax + b$
    * $a$ = the slope of the function
    * $b$ = the intercept of the function (this is the value of the function at $x = 0$)
  - Linear functions have a constant slope (i.e. the slope at any point on the function is equal to $a$)
  - These functions look like straight lines when graphed.
  - e.g., $y = 2x + 3$:  

- **Quadratic functions**
  - These functions are of the form: $y = ax^2 + bx + c$
  - These functions have a parabolic shape when graphed
  - e.g., $y = 2x^2 + 3$:

- **Logarithmic functions**
  - We’ll use the natural logarithm: e.g., $y = \ln(x)$
    * $\ln$ stands for natural log
    * The natural logarithm has a “base” of $e$
    * $e$ is the “mathematical constant”
    * $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n = 2.171828...$ (its an irrational number)
  - The logarithm function has the following properties:
    * $\ln(e) = 1$ (this is how the natural log is defined - it has a base of $e$)
    * $\ln(xy) = \ln(x) + \ln(y)$
    * $\ln(x^y) = y\ln(x)$
    * $\ln(x^y)$ is not defined for $x \leq 0$
- Plotting \( y = \ln(x) \) we can see the shape of the logarithm function:

![Graph of \( f(x) = \ln(x) \)]

Changes and rates of change:
- We will denote “the change in \( x \)” as \( \Delta x \)
  - Think the difference starts with “d”, so we’ll use the Greek letter \( d \), \( \Delta \), to denote change
- Typically, we’ll look at small changes in \( x \). We’ll call this the \textbf{marginal} change.
- To get the rate of change in a function with respect to a change in it’s argument, we do the following:
  - Let \( y = f(x) \), then the rate of change in \( y \) w.r.t. a change in \( x \) is:
    \[
    \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\text{rise}}{\text{run}}
    \]
  - So the rate of change of a function is its \textbf{slope}.
  - e.g., The rate of change with a linear function is:
    \[
    \frac{\Delta y}{\Delta x} = \frac{a + b(x + \Delta x) - a - bx}{\Delta x} = \frac{bx + b\Delta x - bx}{\Delta x} = \frac{b\Delta x}{\Delta x} = b
    \]
    - So the rate of change (or the slope) doesn’t depend on \( x \)
    - i.e., it’s constant (the same everywhere)
  - e.g., The rate of change for a quadratic function is:
    \[
    \frac{\Delta y}{\Delta x} = \frac{a(x + \Delta x)^2 + b(x + \Delta x) + c - ax^2 - bx - c}{\Delta x} = \frac{ax^2 + 2ax\Delta x + (\Delta x)^2 + bx + b\Delta x + c - ax^2 - bx - c}{\Delta x} = \frac{2ax\Delta x + a(\Delta x)^2 + b\Delta x + c}{\Delta x} = \frac{2ax + a\Delta x + b}{\Delta x}
    \]
Here, the rate of change depends on $x$ and $\Delta x$

If $\Delta x \approx 0$, then $\frac{\Delta y}{\Delta x} = 2ax + b$

Here, the rate of change does depend on $x$.

- Note that a tangent to a function is a straight line that has the same slope as the function of interest at some point $x$
  - The point of the tangent is called the point of tangency.
  - e.g.,

\[
\begin{align*}
\text{Derivatives:} & \\
\text{• } A \text{ derivative of a function, } y = f(x) \text{ is defined as:} & \\
\frac{df(x)}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
\text{− i.e., how does the function change for a very small change in its argument?} & \\
\text{− the “d” in } \frac{df(x)}{dx} \text{ is short for “delta” - recall how we used } \Delta \text{ to denote change.} & \\
\text{• Derivatives are rates of change.} & \\
\text{• e.g., For a linear function, } y = ax + b: & \\
\frac{df(x)}{dx} &= \lim_{\Delta x \to 0} \frac{a(x + \Delta x) + b - ax - b}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{ax + a\Delta x + b - ax - b}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} a \\
&= a
\end{align*}
\]
e.g., For a quadratic function, $y = ax^2 + bx + c$:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{a(x + \Delta x)^2 + b(x + \Delta x) + c - ax^2 - bx - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ax^2 + 2ax\Delta x + (\Delta x)^2 + b(x + \Delta x) + c - ax^2 - bx - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2a\Delta x + a(\Delta x)^2 + b\Delta x + c - ax^2 - bx - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2ax + b$$

Second derivatives:

- A second derivative is just the derivative of the derivative.
  
  - 1st derivative = $df(x)$
  
  - 2nd derivative = $\frac{d^2f(x)}{dx^2} = \frac{d}{dx} \left( \frac{df(x)}{dx} \right)$

- The second derivative measures the curvature of a function.
  
  - The curvature is the rate of change in the slope of the function.
  
  - If the second derivative is $> 0$, then the function is convex
    
    * A convex function is one where the slope is increasing as its argument increases
    
    * e.g.,

  - If the second derivative is $< 0$, then the function is concave
    
    * A concave function is one where the slope is decreasing as its argument increases
    
    * e.g.,
• Examples:
  - 2nd derivative of a linear function, \( y = 2x \)
    * 1st derivative: \( \frac{df}{dx} = \frac{d(2x)}{dx} = 2 \)
    * 2nd derivative: \( \frac{d^2f(x)}{dx^2} = \frac{d(2)}{dx} = 0 \)
    * 2nd derivative is zero - saying slope doesn’t increase or decrease with \( x \)
    * This makes sense - a linear function is a straight line - neither concave or convex
  - 2nd derivative of quadratic function, \( y = x^2 \)
    * 1st derivative: \( \frac{df}{dx} = \frac{d(x^2)}{dx} = 2x \)
    * 2nd derivative: \( \frac{d^2f(x)}{dx^2} = \frac{d(2x)}{dx} = 2 \)
    * 2nd derivative is \( > 0 \) - saying slope increases as \( x \) increase
    * Thus this quadratic function is a convex function.

• Points of notation:
  - We denote the 1st derivative with \( f'(x) \). i.e., \( f'(x) = \frac{df}{dx} \)
  - We denote the 2nd derivative with \( f''(x) \). i.e., \( f''(x) = \frac{d^2f(x)}{dx^2} = \frac{df'}{dx} \)
  - We can write higher order derivatives this way as well, e.g. \( f'''(x) \)

Rules of Derivatives:
• It is helpful to know the rules for derivatives
  - You don’t want to have derive the derivative by taking the limit of the rate of change for all functions.
• Fortunately, this means you just have to memorize a few rules for the derivatives of various functions:
  1. Constant function: \( \frac{d\alpha}{dx} = 0 \), where \( \alpha \) is any constant
  2. Scalar multiple: \( \frac{d(af(x))}{dx} = \alpha \frac{df(x)}{dx} = \alpha f'(x) \) where \( \alpha \) is any constant
  3. Sum: \( \frac{d(f(x)+g(x))}{dx} = f'(x) + g'(x) \)
  4. Difference: \( \frac{d(f(x)-g(x))}{dx} = f'(x) - g'(x) \)
  5. Power rule: \( \frac{dx^\alpha}{dx} = \alpha x^{\alpha-1} \), where \( \alpha \) is any constant
  6. Product rule: \( \frac{df(x)g(x)}{dx} = f'(x)g(x) + f(x)g'(x) \)
7. Quotient rule: \( \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \)

8. Chain rule: \( \frac{d}{dx} (f(g(x))) = f'(g(x))g'(x) \)
   - I remember this as it being the “derivative of the outside” (i.e., \( f'(g(x)) \)) times the “derivative of the inside” (i.e., \( g'(x) \)).
   - Also think about it in terms of \( y = f(g(x)) \). If we want to find how \( y \) changes as \( x \) changes we need to do: \( \frac{dy}{dx} = \frac{dy}{dg(x)} \frac{dg(x)}{dx} \)

9. Log functions: \( \frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} f'(x) \)

10. Exponential functions: \( \frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x) \)

Partial derivatives:
   • A partial derivative tells us how a function of many variables changes as just one of its arguments changes.
   \[
   \frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}
   \]
   - Note that the partial derivative symbol is \( \partial \). It is derived from \( \delta \) (delta).
   • Note how the other variables, in this case \( x_2 \), are held fixed when we are looking at the partial derivative with respect to \( x_1 \)
   • Partial derivatives follow all the same rules of derivatives. One just needs to remember to treat the variables we aren’t considering the change in as constants.
   • Examples:
     - \( y = f(x_1, x_2) = x_1^2 x_2 \)
       \[
       \frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{d}{dx_1} x_1^2 x_2 + x_1^2 \frac{dx_2}{dx_1} \]
       - using product rule
       \[
       = 2x_1 x_2 + 0 = 2x_1 x_2
       \]

Optimization
   • We’ll be using differential calculus to help us solve the optimization problems of economic actors (e.g., individuals, households, firms)
   • How this works: suppose you want to find the max of the function \( y = -x^2 + 5 \):
• The maximum/minimum of a function must be where the slope = 0
  − Else one could increase y by moving left or right along the x-axis.
• The maximum will be at a point where the slope is decreasing in x
  − i.e., the 2nd derivative ≤ 0
  − Why? If the slope were increasing then one could get a higher y by increasing x
• The minimum will be at a point where the slope is increasing in x
  − i.e., the 2nd derivative ≥ 0
  − Why? If the slope were decreasing then one could get a lower y by increasing x
• So to find a maximum (or minimum) we have two conditions we need to check. For \( x^* \) to be the point at which the function \( y = f(x) \) is at a maximum, we need:
  1. \( \frac{df(x^*)}{dx} = 0 \) → this is called the “first order condition”
  2. \( \frac{d^2f(x^*)}{dx^2} \leq 0 \) → this is called the “second order condition”
    − For a minimum, the second order condition is \( \frac{d^2f(x^*)}{dx^2} \geq 0 \)
• e.g., if we want to find the maximum of \( y = -x^2 + 5 \) we find:
  − The first order condition: \( \frac{df(x^*)}{dx} = -2x = 0 \implies x^* = 0 \)
  − The second order condition:
    \[ \frac{d^2f(x^*)}{dx^2} = -2 < 0 \implies \text{This is a max (not a min)} \]
• Optimization of functions with multiple arguments.
  − We will have as many first order conditions (FOCs) as we have arguments.
  − e.g., For \( y = f(x_1, x_2, ..., x_n) \) we have \( n \) FOCs:
    \[ \frac{\partial y}{\partial x_1} = 0 \]
    \[ \frac{\partial y}{\partial x_2} = 0 \]
    ...
    \[ \frac{\partial y}{\partial x_n} = 0 \]
There are second order conditions too, but they get pretty complicated and we’ll not worry about that here.

**Useful algebraic facts:**

- **Rules of exponents:**
  - \( x^{-n} = \frac{1}{x^n} \)
  - \( x^n x^m = x^{n+m} \)
  - \( (x^n)^m = x^{nm} \)
  - \( \frac{x^n}{x^m} = x^{n-m} \)

- **Solving systems of equations:**
  - To solve, you need the # of equations= # of unknown variables
  - Even if # of equations= # of unknowns, you may not be able to solve
    - Yes, if equations are linear
    - Not necessarily if equations are non-linear (may have no solution or multiple solutions)

**Microeconomics Review**

**What is economics?**

- Economics is about the allocation of scarce resources.
- Economics is a social science. It is the social science.
- How are resources allocated? There are three ways:
  1. Love
  2. Force
  3. Trade

- We’ll focus on the latter - that is, on market mechanisms for allocating resources (although economists may talk about all three).

**Models:**

- Economists use **models** to understand economic phenomena.
- Models are representations of reality. They simplify reality.
- We will formalize models with mathematical equations and/or graphical representations.
  - This allows us to be clear what we are stating with our model.
  - It also allow us to use the tools of mathematics to analyze the phenomena we model.
- Models will relate **endogenous variables** to **exogenous variables**.
  - Endogenous variables are those determined within the model.
  - Exogenous variables are those determined outside the model.
- We make two assumptions about how economic agents act in our models:
1. **The optimization principle**: Agents act to make themselves as well off as possible.

2. **The equilibrium principle**: Endogenous variables adjust until the model is “balanced” (or comes to a steady point)
   - There are many equilibrium concepts. The one we’ll see the most is that supply and demand are balanced.
   - Equilibrium is the state models tend to given our assumption of optimization and and incentive structure in the model.
   - Equilibrium is a way to make sure that economic actions/changes determined by the model are consistent with each other.

A model of supply and demand:

- The demand curve:
  - A demand curve traces out the **willingness to pay** (or **reservation price**) of individuals in the market.
  - Draw a demand schedule...
  - Show how we can smooth out the demand schedule

- The supply curve:
  - A supply curve shows the number of units that will be supplied at any given price
  - Draw a supply schedule that is vertical - i.e. fixed at any price

- Market equilibrium:
  - If markets are competitive, the equilibrium will be determined as the price and quantity where supply equals demand
  - Draw supply and demand together and note equilibrium...
  - Competitive means that information is known by all, no purchaser or seller can affect the price by buying more or less.
  - Note that the market tends towards equilibrium
    - Consider what happens if the price were higher than the eq’m price...
    - Consider what happens if the price were lower than the eq’m price...

Four market scenarios:

1. Competitive markets
   - This is what we examined above.
   - Prices move to “clear the market” (i.e., balance supply and demand)
   - The resulting allocation:
     - Everyone willing to pay an amount above the market price gets the good/service.
     - No one who is willing to pay below the market price gets the good/service.

2. A discriminating monopolist
   - Consider one seller who knows the willingness to pay of each buyer and can charge a different price to each
• What would this seller to do maximize profit? Yes - charge each buyer the maximum she is willing to pay
• Buyers are happy - they get something for what they are willing to pay
• Sellers really benefit - they can extract the maximum from each buyer
• The resulting allocation:
  – Everyone willing to pay an amount above the market price gets the good/service.
  – No one who is willing to pay below the market price gets the good/service.

3. An ordinary monopolist
• Consider one seller who can control the price of the good/service, but must charge all potential buyers the same price
• What would this seller to do maximize profit?
  – Note that total demand is a function of price - call this \( D(p) \)
  – Demand decreases in price
  – So the monopolist is going to charge a price that weights the benefits of more per unit sold with reductions in units sold
  – Generally, this results in a price that is higher than the competitive market price.
  – Why? Because in the competitive market, the seller doesn’t affect the price, and so doesn’t take into account reducing units sold can increase prices
• The resulting allocation:
  – Not everyone willing to pay an amount above the market price gets the good/service. Only those willing to pay at least the monopolist’s price get it (and that price is generally higher than the market price).
  – Supply is restricted as compared to the competitive market.

4. A competitive market with a price ceiling
• A limit on the highest price sellers can offer.
• Note that this price ceiling only matters to the extent that it constrains the market price (i.e., it only matters if the market equilibrium would result in a price higher than the price ceiling).
• What happens to the allocation of goods/services?
  – If the price ceiling “binds”, then at that price ceiling, demand exceeds supply.
  – This is called a shortage.
  – What this means is that at that price, there are more people who would like the good than there are goods to be sold.
  – Thus market prices aren’t going to be used to determine the allocation of goods. The shortage would normally push prices up, but the price ceiling is constraining them.
• The resulting allocation:
  – Not everyone willing to pay an amount above the price ceiling gets the good/service. There isn’t enough supply at this price to satisfy demand.
  – We don’t know how the allocation is distributed among those willing to pay an amount at or above the price ceiling.
  – The allocation will be determined by things outside the of our model of the market - e.g., who is friends with whom, who is willing to pay time costs to queue up for the good, etc.

How do we determine which allocations are best?
• We need to take into account everyone - buyers and sellers.
• A useful concept: **Pareto efficiency**
  
  – An allocation is Pareto efficient (or Pareto optimal) if one cannot find a different allocation that makes at least one person better off and no one worse off.
  
  – A change in the allocation that can make at least one person better off and no one worse off is called a **Pareto improvement**.

• Pareto efficiency means that all “gains from trade” are exhausted. Why?

  – Two people trade only if at least one of them will be made better off and no one worse off.
  
  – So if all gains from trade are exhausted, then we must have a Pareto efficient outcome (if we didn’t, then there would exist a trade that would make at least one party better off and no one worse off - but, by definition, this can’t be the case if all gains from trade have been exhausted).

• What about the Pareto efficiency of the market allocations described above?

  1. Competitive Market

     – Pareto efficient

     – Note that everyone with the good is willing to pay more than anyone who doesn’t have the good - so there is not room to make someone at better off and no one worse off with a different allocation.

  2. Discriminating Monopolist

     – Pareto efficient (remember, same allocation as competitive market)

     – Note that distribution of income may be way different than competitive outcome - but both efficient.

  3. Ordinary Monopolist

     – Not Pareto efficient

     – Consider that there is a mutually beneficial trade between the monopolist and a potential buyer if that monopolist could charge a lower price to just that buyer. But since she can’t that trade can’t happen.

  4. Price control case

     – Not Pareto efficient

     – There are trades that could be mutually beneficial, but are disallowed by the price control