Notes on Overlapping Generations Models

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1 Introductory Comments and Overview

These notes are on a series of overlapping generations models. These are dynamic equilibrium models and thus incorporate both dynamic optimization at the individual level and market clearing conditions.

The individual optimization aspect of these models was covered in the notes on two period optimization. The lecture on competitive equilibrium in static economies sets the stage for the equilibrium aspect of the models summarized in these notes.

2 Simple OG model: Shell(*JPE*, 1971)

We start with the overlapping generations model from Shell. This is an easy paper to read though be careful in thinking about the structure of markets for the individuals in the economy.

2.1 Assumptions

- infinite horizon; \( t = 1, 2, \ldots \).

- Agents live for two periods – youth and old age and then they die!

- One person, normalized, born each period. So that total population is 2. No population growth.

- Assume price taking behavior

- Individual born in period \( t \) is a member of generation \( t \).

- Endowments: One unit of chocolate each period.

- Preferences of generation \( t \): Indifferent between goods over time so that

\[
\begin{align*}
  u_t(c_t^t, c_{t+1}^t) &= c_t^t + c_{t+1}^t. \quad (1)
\end{align*}
\]
With these preferences, consumption today and tomorrow are perfect substitutes. Note the notation in (1): superscripts will refer to the generation and subscripts will refer to the time period.

Time starts with period one so we also need to assume that there is an initial old individual endowed with one unit of chocolate as well. This initial old generation meets generation 1 in period 1.

We are interested in a competitive equilibrium in which all buyers and sellers take prices as given. An equilibrium has two components: individual optimization and a consistency requirement (market clearing in this case) across the choices of the individuals. We look at these in turn.

### 2.2 Individual optimization

Agents maximize utility subject to a budget constraint taking prices as given. That is, agents do not consider whether their demands are consistent with supply and market clearing: they simply maximize given their budget constraints.

We let $p_t$ be the price of period $t$ chocolate in terms of period 1 chocolate. So these are present value prices in that the period one commodity is taken as the numeraire.

The budget constraint of a (representative) generation $t$ agent is

$$p_t c_t^t + p_{t+1} c_{t+1}^{t+1} = p_t + p_{t+1}.$$  (2)

The right side of this constraint is the total value of the endowment of a generation $t$ agent. The left side is the level of spending over the two periods in which the generation $t$ agent obtains utility from consuming, periods $t$ and $t+1$.\footnote{In this market structure, the agent could buy goods in any period but the agent only gets utility from consuming when alive.}

With these preferences, the individual will wish to consume only in period $t$ if $p_t < p_{t+1}$ and will wish to consume only in period $t+1$ if $p_t > p_{t+1}$. If $p_t = p_{t+1}$, then the individual will be indifferent about the timing of consumption.

### 2.3 Market Clearing

In addition to utility maximization, there is a second condition for equilibrium: market clearing. This means that agents demand’s generated by their optimization problem need to sum to the supply of goods in the market in each period.

Recall that in Shell’s paper, agents all meet in time period 0. They trade promises to deliver goods in some future period. Thus all markets open and clear in period 0.

Market clearing conditions are given by,

$$c_{t-1}^t + c_t^t = 2$$  (3)
for \( t = 1, 2, \ldots \). Remember that \( c^t_{t-1} \) is the period \( t \) consumption of generation \( t - 1 \).

Note that equilibrium does not refer to a single period, but the whole path of the economy – i.e. it requires a statement of a price sequence, \( p_1, p_2, p_3, \ldots \). This is implicit in the individual’s budget constraint since future prices determine what is affordable.

A sequence of prices, \( \{p_t\}_{t=1}^\infty \), is called an equilibrium if the demands of agents induced by this sequence satisfies market clearing for all time periods. Now look at Shell’s simple example to find an equilibrium.

**The sequences** \( p_t = 1 \) and \( c^t_t = c^{t+1}_t = 1 \) **for all** \( t \) **is an equilibrium.** This equilibrium is called autarky since agents consume their endowments in each period of life. To see why this is an equilibrium, we check individual optimization and market clearing.

Faced with these prices, agents do not care about the timing of their consumption. Hence, \( c^t_t = c^{t+1}_t = 1 \) for all \( t \) solves the agent’s maximization problem. This choice of consumption also satisfies the condition for market clearing. So an allocation in which each agent consumes his/her endowment in each period is an equilibrium.

If \( p_t = 1 \) for all \( t \), the requirements of market clearing and optimization are satisfied. In fact, since the initial old cannot trade (that is they have nothing to offer the young in exchange for goods) and therefore must eat their endowment, it is easy to see that autarky is the only competitive equilibrium in this economy.

### 2.4 Welfare analysis

We have found an equilibrium. Now we look at the allocations that the planner can achieve. To do so, we must specify what is feasible for the planner and what is desirable. For this simple economy, feasible allocations must satisfy:

\[
c^{t+1}_{t+1} + c^{t+1}_{t+1} = 2
\]

for \( t = 1, 2, 3, \ldots \).

That is, the planner faces the same resource constraints in this economy as we saw in the market clearing condition. The planner cannot create goods out of thin air! Note that this condition holds for each period across generations.

In the competitive equilibrium we have just described, agents eat their own endowments. Is that a Pareto Optimal allocation?\(^2\) Or is there another allocation that makes some agents strictly better off and no agents worse off? So, can the planner dominate (using Pareto optimality as a criterium) this allocation?

\(^2\)We say that allocation \( x \) Pareto dominates allocation \( y \) if allocation \( y \) is better for some agents and not worse for any of them relative to \( x \). For Shell’s economy, an allocation is a feasible distribution of the total endowment in each period.
Shell’s answer is **YES**. Suppose that the planner gives the endowment of the first young guy to the old. The old guy is better off while the young guys loses a unit of utility. In the next period, transfer a unit from the new young to the old. So the first young guy has now been compensated for the initial transfer. Keep on compensating all through time and we obtain an allocation which Pareto dominates the original equilibrium.

Note that this seems to contradict the First Fundamental Welfare Theorem, which states (under some conditions) that competitive equilibria are Pareto optimal. Why is this so? Some people viewed this as a consequence of the implicit incomplete market structure in the OG setting. That is, agents cannot trade on markets before they are born. Shell says not true. His equilibrium corresponds to the complete market participation one as well.

The key is the double infinity of people and goods. If the economy had a finite life, then this transfer scheme would not work and the autarkic competitive equilibrium allocation would be Pareto optimal.

Does this result of the sub-optimality of competitive equilibria always work or are there some conditions implicit here? Note that we used the perfect substitutes in consumption over time to be able to see exactly what we had to do to compensate the old agents for their lost endowment in youth. So clearly intertemporal preferences matter. Second, we had to have some restriction on the population growth to make these transfers feasible. Would we have been able to do this if the population had been decreasing? See the exercises later in this section and the discussion of the Gale paper to follow.

So more generally, we must search for conditions on preferences and population growth to see whether these schemes are possible. Now for preferences, we will care about the marginal rates of substitution over time at a particular point. This, we will see, is really a real interest rate.

### 2.5 Role for Money

**Given** that there exist Pareto dominant allocations, how can we support them? The initial young will not give the initial old the transfer effected by the planner because they have no claims on the endowment of the next generation. In thinking of Pareto domination, we should also think about the introduction of "institutions" which can help to achieve desirable outcomes. A natural one in this setting is some type of claim, such as money.

Shell says, suppose that the initial old use their chocolate wrappers as a form of fiat money. So that there is now a single piece of money in the economy. Notice that this is not a commodity money in that the wrappers, per se, have no value – try eating one! This is an example of fiat money.

Suppose that we let $\pi_t$ be the price of goods in terms of money in period $t$. I.e. $\pi_t$ tells us how many pieces of money it takes to buy a unit of the good in period $t$. As individuals can hold money
over time their budget constraints can be written as

\[ \pi_t c_t^t + m_t = \pi_t \]  \hspace{1cm} (5)

and

\[ \pi_{t+1} c_{t+1}^t = \pi_{t+1} + m_t. \]  \hspace{1cm} (6)

These constraints are in money terms not goods.

Here \( m_t \) is the money held by a representative generation \( t \) agent. The money is purchased by exchanging goods for money in period \( t \) and then sold for goods in the next period at a price of \( \pi_{t+1} \). With money it is often easier to think of a sequence of trades: sell goods in youth for money and then buy goods with money in old age.

Now we need to find an equilibrium for the monetary economy, \( \pi_t \) for \( t = 1, 2, ..., \). Shell says that an equilibrium occurs when \( \pi_t = 1 \) for all \( t \). At these prices, agents in generation \( t = 1, 2, ... \) still don’t care about the timing of their consumption. Their budget constraint permits any combination of consumption in youth and old age that sums to 2. They also don’t care about how much money they hold: \( m_t \) disappears from the budget constraint. So we can use this indifference to construct an equilibrium in which young agents sell their endowments for money – i.e. give up their piece of chocolate for the wrapper. When old they sell this wrapper to the next generation at a price of 1. Since the initial old received an extra unit, this is the Pareto dominant allocation described before!

We introduced a fiat money and this thing had value which allowed us to support a desirable allocation. If the economy was finite, the whole thing would unravel – money would become a ”hot potato” in that no one would want to hold it. In order for money to have value, need an infinite horizon.

For the Shell economy, the competitive equilibrium without money is not Pareto optimal and the introduction of money, if it is valued, can restore Pareto optimality. Again, what is special and what is general? To see, we go on to consider more general models of production and exchange.

<table>
<thead>
<tr>
<th>Exercise</th>
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<tbody>
<tr>
<td>1. A good exercise would be to redo the analysis of the Shell economy allowing population growth.</td>
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<tr>
<td>2. Suppose that the utility function was</td>
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\[ u_t(c_t^t, c_{t+1}^t) = c_t^t + \beta c_{t+1}^t. \]  \hspace{1cm} (7)

What is the equilibrium in Shell’s economy without money? When is the equilibrium Pareto optimal?

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3As we shall see, there may be other equilibria. In particular, there is generally an equilibrium when fiat money has no value.
3 General Model of Exchange: Gale (*JET*, 1973)

Now lets look more thoroughly at the OG model with savings. The paper by Gale provides a very nice clean presentation.\(^4\) We now generalize the environment of Shell to allow for population growth and a more general endowment pattern.

3.1 Basic Assumptions

The notation here follows that in the Gale paper which, of course, differs from that used by Shell. Here are the basic pieces of Gale’s model:

- Generation of size \(N_t\) born each period. Assume that \(N_t = \gamma N_{t-1}\). So that \(\gamma\) is the rate of population growth.

- Individuals live for two periods and are all identical within a generation

- Endowments: \(e = (e_0, e_1)\) in the two periods of life

- Consumption: \(c(t) = (c_0(t), c_1(t+1))\) is the consumption profile for an agent born in period \(t\).

- Individual preferences are given by a utility function \(U(c)\) which is increasing and concave. Preferred sets are strictly convex.

For this economy, we will study the competitive equilibrium and then determine its welfare properties.

3.2 Competitive Equilibrium

We can assume the same market structure as specified by Shell. To study a competitive equilibrium we first consider individual optimization and then look at market clearing. We will argue that autarky is a competitive equilibrium for this economy.

3.2.1 Household Optimization

To study the competitive equilibrium, we need to specific the budget constraint for a representation household of generation \(t, t = 1, 2, 3, \ldots\) agent. Letting \(p_t\) be the price of period \(t\) consumption goods in terms of the numeraire, the budget constraint for generation \(t\) agents is

\[
p_t c_0(t) + p_{t+1} c_1(t+1) = p_t e_0 + p_{t+1} e_1. \tag{8}
\]

Dividing both sides by \(p_t\) and letting \(\rho_t = \frac{p_t}{p_{t+1}}\), we can rewrite the budget constraint as

\[
\rho_t (e_0 - c_0) + (e_1 - c_1(t + 1)) = 0. \tag{9}
\]

\(^4\)We stress the analysis in Section II of that paper.
Think of $\rho_t$ as a real interest rate since it tells us how much consumption we have tomorrow from saving a unit today. You can graph the budget constraint in the usual way here. The slope of the budget constraint is simply $-\rho_t$.

The individual choice problem is then to maximize utility subject to the budget constraint, (9). This will imply a policy function, $c^*(\rho_t) = (c_0^*(\rho_t), c_1^*(\rho_t))$ since the entire consumption decision depends on $\rho_t$. From this emerges a savings decision, $s^*(\rho_t) = e_0 - c_0^* (\rho_t)$.

### 3.3 Market Clearing

In each period, the market for consumption goods must clear. That is the supply of goods must equal to the total demand. Since there are $N_t$ young and $N_{t-1}$ old agents in period $t$, the total supply is $N_t e_0 + N_{t-1} e_1$. Using the notation of optimal consumption choices, total demand is $N_t c_0^*(\rho_t) + N_{t-1} c_1^*(\rho_{t-1})$. Thus market clearing in period $t$ is given by

$$N_t e_0 + N_{t-1} e_1 = N_t c_0^* + N_{t-1} c_1^*$$

Using $N_t = \gamma N_{t-1}$, this condition is

$$\gamma (e_0 - c_0^*) + (e_1 - c_1^*) = 0.$$

### 3.4 Steady State Equilibrium

Gale focuses on steady state allocations where the consumption profile of generation $t$, $c^*(t) = (c_0^*(t), c_1^*(t + 1))$ is independent of $t$: i.e. while consumption in youth and old age may be different, these consumption levels are the same across generations. Note that in a stationary competitive equilibrium, $\rho_t$ will be time independent as well.

In this case, subtracting the conditions of feasibility from that of the budget constraint implies that the stationary competitive equilibria must satisfy

$$(\rho - \gamma)(e_0 - c_0) = 0.$$  

There are two endogenous variables in this condition. The first is $\rho$, the intertemporal terms of trade in a steady state and the second is $c_0$, the level of consumption in youth in a steady state.

From (12), there are at most two possible steady states. The first, Case I, is where the population growth rate equals the interest rate. This is an optimal allocation in that it is the best steady state allocation given the resource constraint. That is, if we restrict a planner to steady state allocations, this is the allocation which maximizes $U(c)$. Denote this allocation by $\bar{c} = \bar{c}_0, \bar{c}_1$.

In the second steady state, consumption equals the endowment in each period of life. Call the second one, Case II, autarky.
We will concentrate on autarky and its welfare properties. Then we discuss the non-autarkic steady state allocation, which as Gale argues has certain optimality properties, and how to support it.

3.5 Autarky

Here we characterize the autarkic outcome and its welfare properties.

3.5.1 The Autarkic Steady State

One of the two steady states is autarky, where household’s consume their endowment: $e_0 = \check{c}_0, e_1 = \check{c}_1$ for all generations and the real interest rate is $\check{\rho}$.

By construction, $\check{\rho} = \frac{U_1(e_0, e_1)}{U_2(e_0, e_1)}$. In words, the real interest rate supporting the autarkic steady state equals the marginal rate of substitution at autarky. Thus, faced with intertemporal terms of trade of $\check{\rho}$, it is optimal for households to consume their endowment.

To be an equilibrium, it must be the case that autarky is market clearing. But this is immediately true since if each agent consumes his/her endowment in a period, then total consumption must equal the total endowment, satisfying (11).

3.5.2 Welfare Properties

Is the autarkic solution Pareto Optimal? Answering this question is one of the key contributions of this paper. Here we are going beyond the analysis in Shell since we have more general preferences and population growth.

To answer this question, look at an individual agents indifference map in terms of consumption in youth and old age. The budget constraint which supports the choice of autarky has slope of $-\check{\rho}$ while the set of feasible allocations is bounded by the line with slope $-\gamma$.

The key contribution in this paper is to demonstrate that the Pareto optimality of autarky depend on whether or not $\check{\rho}$ exceeds $\gamma$.

Gale distinguishes two cases: the Samuelson case and the Classical case. The Classical case occurs when $\check{\rho}$ exceeds $\gamma$. The Samuelson case occurs when $\check{\rho}$ is less than $\gamma$. So when the marginal rate of substitution at the endowment point is higher (lower) than the population growth rate, then the economy is in the Classical (Samuelson) case.

The interest in these two cases is that they provide the answer to the question about the optimality of the autarkic steady state.

THEOREM (Gale, Thm 3) The no-trade equilibrium is Pareto optimal in the Classical case and not in the Samuelson case.

Proof.

5 The paper distinguishes these cases in two ways. Here we focus on the distinction in Theorem 2 of Gale.
(i) Suppose that we are in the Samuelson case. Consider a feasible redistribution away from young to old agents, going from autarky to the optimal steady state, $\bar{c}$. The initial old are made better off while the young have a higher lifetime utility as well.

Locally, redistributions from young to old at a rate $\gamma$ are feasible and increase utility, relative to autarky, since $\gamma > \tilde{\rho}$. So autarky is not Pareto optimal.

ii) Suppose we are in the classical case and suppose that there exists a non-autarkic allocation, $x(t)$, which is weakly preferred to autarky by all agents. So, for this allocation to be at least as good as autarky, it must not be strictly affordable: $\tilde{\rho}(e_0 - x_0(t)) + (e_1 - x_1(t + 1)) \leq 0$ for all $t$.

Since $x(t)$ is feasible, we know that $x_1(t + 1) = \gamma(e_0 - x_0(t + 1)) + e_1$. Substituting this into the above expression $\tilde{\rho}(e_0 - x_0(t)) \leq \gamma(e_0 - x_0(t + 1))$.

Without loss of generality, assume that $x_0$ not equal to $e_0$ for the first generation. Hence, $(e_0 - x_0(t + 1)) \geq (\tilde{\rho})t(e_0 - x_0(1))$.

Now as time goes on the right hand side becomes unbounded since $\tilde{\rho}$ exceeds $\gamma$. But the left side is bounded since both the endowment and consumption are bounded. So this gives us a contradiction.

Note that one can think about the chocolate example of Shell. Pass back chocolate to the old. Then the young need to be reimbursed enough so that they are indifferent. Now the tax on the next generation, per person, depends on the relative size of the populations and the MRS of the initial young –the latter determines how much they must be repaid. Now if the population is growing fast enough the amount each succeeding young generation must pay falls so that the whole redistribution scheme is feasible. But if the population growth is too slow, then the tax may drive consumption negative. This is the intuition of the proof.

3.6 Role of money

Suppose we are in the Samuelson case in which the autarkic equilibrium is not Pareto optimal. How can we achieve the planner’s outcome? It requires that the first young generation give up something to the current old. Why should the young do this in a decentralized economy? The answer is that if someone creates money, as in Shell’s story, then there is a way for generations to be linked. So again money serves to support intertemporal trades underlying optimal equilibria.

4 Simple Production Economy

Now look at a production economy rather than an exchange model since this will bring us closer to some of the structure we will need in the next section of the course. Want to be more explicit about the money side of these models. So far we have been asking questions mainly about the autarkic equilibrium – is it optimal or not? Now lets focus on the role of money and the behavior of the
economy out of steady state. Find a lot of rich things going on here in terms of dynamics.

4.1 Fiat money

Fiat money is intrinsically useless. This was a property of Shell’s chocolate wrappers and is true for fiat money in general. The point is that the pieces of paper that we use for exchange purposes have no value in and of themselves. Try to eat a dollar bill. A second important part of fiat money is its inconvertability. That is, no one is standing ready to convert the $ into anything of direct utility – like gold or chocolate. Its value is determined in the market.

So, the value of $ is really derived from the fact that others value the bills as well. As Wallace stresses, this makes equilibria in which fiat money has value tenuous – i.e. likely to fall apart. All it takes is a loss of confidence in the fiat money and it can lose its value. This loss of confidence may arise from government devaluing the currency thru seignorage or just the beliefs of agents themselves.

4.2 Model

This model is frequently used as a basic structure for asking questions in dynamic macroeconomics. Remember that a good model is simple enough to work with yet captures an important element of investigation. Here we have simplicity and richness at the same time. First go thru the setup to convince you of the simplicity and then move on to the questions of richness.

4.2.1 Assumptions

- Generation of size N born each period
- Agents live for 2 periods and then die
- time starts at period 1 and goes to infinity: \( t = 1, 2, 3, \ldots \)
- At \( t = 1 \), there is an initial old generation endowed with units of money per capita. (So we assume that fiat money already exists, is it valued?)
- Agents are endowed with a unit of leisure time in youth and nothing else
- Agents consume leisure in youth and consume some of the single good in old age
- Agents supply labor, \( n \), in youth and hold money balances over time to finance consumption in old age
- Preferences are given by: \( U(c_{t+1}) - g(n_t) \). Here \( U(\cdot) \) is strictly increasing and concave and represents the utility from consumption in old age. The function \( g(\cdot) \) is strictly increasing and convex and represents the disutility of working in youth. Note that these preferences are time
separable: the marginal utility of what consumption in period \( t + 1 \) is independent of labor input in period \( t \).

- Technology: We will assume there is a simple linear technology here which converts labor input into output: \( y_t = n_t \).

### 4.3 Planning solution

Suppose that you are the planner. Your objective might be to maximize the utility of each agent. Since they are all identical you may treat them equally. So consider a planner choosing a level of consumption and a level of work each period for agents which maximizes the utility of a representative agent.

Since there is no physical store of value, i.e. there are no inventories and no capital, output produced must equal the consumption of the old. So the planner’s problem has no dynamics and thus, using the equal treatment property, simplifies to maximize \( U(c) - g(n) \) subject to \( c = y = n \) with \( 0 \leq n \leq 1 \). This is the problem we solved when we looked at static general equilibrium. The interior solution, denoted \( n^{**} \), solves

\[
U'(n^{**}) = g'(n^{**}).
\]  

(13)

Note that there is only one value of \( n \) which satisfies this equality given the assumptions made on the curvature of \( U(\cdot) \) and \( g(\cdot) \). We assume \( U'(0) > g'(0) \) and \( U'(1) < g'(1) \) which implies that \( n^{**} > 0 \). Here autarky is not the solution to the planners problem. From the the discussion of Gale’s paper, that we are in the Samuelson case! Remember that \( y^{**} = n^{**} \) from the technology.

### 4.4 Competitive Equilibria

We start with optimization. We then look at market clearing.

### 4.5 Optimization

The period \( t \) price of the good in terms of money will be denoted by \( p_t \). The budget constraint for generation \( t \) (born in period \( t \)) is given by:

\[
m_t = p_t n_t.
\]

(14)

and

\[
p_{t+1} c_{t+1} = m_t
\]

(15)
where the index for the individual agent is suppressed. Thus \( n_t \) is the labor supply of the representative agents in generation \( t \). In equilibrium, the total labor supply of generation \( t \) is \( N_t n_t \).

Agents acquire money balances when young and then convert them into goods in youth. These two budget constraints can be merged into one:

\[
c_{t+1} = \frac{p_t n_t}{p_{t+1}} = \rho_t n_t.
\]  

(16)

Here \( \rho_t \) is a real interest rate as it indicates the return in real goods in period \( t+1 \) of producing and storing an additional unit in period \( t \). However, (16) obscures the fact that here we will assume that agents cannot meet together to trade but that money is essential to provide a connection across generations. This is made explicit in equations (14) and (15).

So from the budget constraint we see two important things. First, money is held as a means of facilitating intertemporal exchange. Here it operates as a medium of exchange. Money can facilitate exchange only because it is a store of value. That is, holding money today yields something (the same amount of money) tomorrow.

Second, the budget constraint contains future prices. Since we are taking the view that agents have perfect foresight. That is, in equilibrium, the price for the future that guides their decisions today is the price that really arises in equilibrium. Alternatives to this would prescribe an arbitrary price for what agents expect in the future.

While we can study decisions under these arbitrary prices, perfect foresight seems more compelling. The point is that without perfect foresight rational agents will consistently be making wrong forecasts.

So agents maximize utility subject to a sequence of budget constraints, taking prices as given. There is also a constraint that \( 0 \geq n_t \geq 1 \). Optimization yields a first order condition for an interior solution:

\[
\rho_t U'(\rho_t n_t) = g'(n_t).
\]

(17)

This first order condition tell us how much agents are willing to work as a function of the intertemporal rate of return from work, \( \rho_t \). The way to interpret \( \rho_t \) is as a real wage. Agents are paid in dollars at the rate of \( p_t \) when young and then purchases commodities at a price \( p_{t+1} \) in old age. So the ratio of these is really their real wage – equivalent here to a real interest factor \((1+\text{the real interest rate})\).

Variations in the real wage will induce agents to alter their decisions on production and consumption just as in the static models. Total differentiation of the first order conditions with respect to \( n_t \) and \( \rho_t \) yields

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6 More formally we could have carried a superscript \( i \) throughout the analysis to indicate agent \( i \) and then used the fact that agents are identical to replace a summation across agents with the product of \( N \) and their common per capita decision.
\[
\frac{dn_t}{d\rho_t} = \frac{U'(\rho_t n_t) + \rho_t n_t U''(\rho_t n_t)}{-[\rho_t^2 U''(\rho_t n_t) - g''(n_t)]} \tag{18}
\]

By our assumptions on \(U(\cdot)\) and \(g(\cdot)\), the denominator in this expression is positive. Thus whether employment increases with the real wage (interest rate). We know from elementary microeconomics that there are two effects operating here. The income effect says that if the real wage increases, then we should consume more of everything, including leisure, so that \(n_t\) would fall. The substitution effect says that leisure has become relatively more expensive as \(\rho_t\) increases so that \(n_t\) should increase with \(\rho_t\). Which of these dominates will depend on the preferences. There are two cases:

- **GROSS SUBSTITUTES**: \(U'(x) + xU''(x) > 0\) or \(R(x) < 1\).
- **GROSS COMPLEMENTS**: \(U'(x) + xU''(x) < 0\) or \(R(x) > 1\).

where \(R(x) \equiv -\frac{xU''(x)}{U'(x)}\) and is a measure of the curvature of \(U(x)\). This is often called the Arrow-Pratt measure of relative risk aversion in discussing choice under uncertainty. That is not quite appropriate here because there is no uncertainty in the model and the utility function is multi-dimensional. So view \(R(\cdot)\) as a curvature measure.

### 4.6 Market Clearing

Market clearing in each period requires that money demand and supply be equated and that the supply and demand for goods be equated. However, these conditions are not independent due to Walras Law. Here, we only need to consider one market clearing condition, the other is implied by the first and the individual’s budget constraint.

Goods market clearing implies that

\[
Nn_t = \frac{Nm}{p_t} \tag{19}
\]

for \(t = 1, 2, 3, \ldots\) where \(N\) is the level of the population and \(m\) is the per capita money supply, originally endowed to the initial old. Obviously the \(N\)’s cancel and we can write the model in per capita terms as in Shell’s framework.

### 4.7 Equilibrium

Equilibrium is a pair of sequences of \(\{p_t, n_t\}\) for \(t = 1, 2, 3, \ldots\) such that

- Markets clear
- \(n_t\) maximizes the utility of a representative generation \(t\) agent given \(p_t\) for \(t = 1, 2, 3, \ldots\)
To find an equilibrium, we usually substitute the decision functions (correspondences are dealt with somewhat differently) of agents into the aggregate excess demand function and then find prices (or prove the existence of some) which clear markets. Here we have a very simple model and can characterize the equilibrium sequences of \( p_t \) by substituting market clearing into the first order conditions for agents. This yields our fundamental equilibrium difference equation of

\[
    n_{t+1} U'(n_{t+1}) = n_t g'(n_t)
\]

for \( t = 1, 2, 3, ... \).

This equation tells us everything about the set of equilibria for this economy. We look at steady state and also non-stationary equilibria.

### 4.7.1 Steady-state equilibria

As suggested by Gale, there are always two steady state equilibria for these types of models.

- **AUTARKY**: \( \rho_t = n_t = y_t = 0 \), \( p_t = \infty \),
- **MONETARY**: \( p_t = p^* \), \( \rho_t = 1 \) and \( n_t = y_t = n^* \) for \( t = 1, 2, 3, ... \).

An autarkic equilibrium always exists since one can consider the allocation as arising when money has no value. This is really like the allocation described by Shell in which money is essentially absent.

A second possible steady state equilibrium occurs when \( p_t = p^* \) for all \( t \) and \( n^* = y^* \) solves \( U'(n^*) = g'(n^*) \).

We know that an \( n^* > 0 \) solving this exists and by our assumptions on the curvature of \( U(\cdot) \) and \( g(\cdot) \), it must be unique. Further, \( n^{**} = n^* \) where \( n^{**} \) is the solution to the planner’s problem. So in the second equilibrium we have a Pareto optimal outcome while in the first we have a sub-optimal equilibrium.

### 4.7.2 Non-steady state equilibria

The virtue of this model is that it eases the analysis of non-steady state equilibria. This is obtained directly from the implicit difference equation given in (20).

Any sequence of \( n_t \) for \( t = 1, 2, 3, ... \) which satisfies this equation is an equilibrium for this model. From this sequences and market clearing, we can then derive the equilibrium prices, \( p_t \) for \( t = 1, 2, 3, ... \). Note again, that it is improper to think of equilibrium in one period – think dynamically since all decisions depend on the future.

To analyze this difference equation, look at its slope:

\[
    \frac{dn_{t+1}}{dn_t} = \frac{n_t g''(n_t) + g'(n_t)}{n_{t+1} U''(n_{t+1}) + U'(n_{t+1})}
\]

The sign of this derivative depends on the sign of the denominator which depends on whether or not we have gross substitutes or complements. We study these cases in turn.
4.8 Dynamics with Gross Substitutes

The graph from class characterizes the dynamics in this case. Since we have assumed the "Samuelson case" the dynamical equation has a slope flatter than unity at the origin. Otherwise, the slope would exceed one at the origin and there would be no monetary equilibria. Note that the curve cuts the 45 degree line at $n^*$. Finally, by our assumption of gross substitutes, the curve has a positive slope in the gross substitutes case.

The set of equilibria is then easily seen by the fact that a difference equation needs a starting condition. That is, we need to specify a level of labor input the first period to get the economy started. Once we do that the dynamics are stipulated by the equation. Note further that specifying $n_1$ is the same as specifying an initial price level $p_1$. So there is a bit of indeterminacy in this model—there is a continuum of equilibria which start from an arbitrary starting value in the interval between 0 and $n^*$. If we start the economy in this interval, the economy goes to the autarkic solution. If we start the economy outside of this interval, the economy explodes.

So we see that the Pareto dominant steady state is not stable. Instead, if the economy starts near $n^*$ but not at it, the economy will either blow up (which is not a feasible solution) or the economy will converge to autarky.

Along the path to autarky, the price level is blowing up since employment and thus output is falling and the money supply is constant. If the price level is getting larger and larger then the real interest rate is falling too so that agents are not working as much. Note that we have inflation here even though the money supply is fixed.

4.9 Dynamics with Gross Complements

We have a different graph as the dynamical equation is negatively sloped. Remember though that the origin is also an equilibrium. Now the dynamics will depend on the shape of the difference curve around $n^*$. If the slope is less than one in absolute value, then the equilibrium is locally stable. If the slope exceeds one than the economy takes off in ever-increasing cycles and we are soon outside of the feasible set for the economy. Since labor supply is downward sloping in this case, it takes a high real interest rate to get people to work a little in period t. But to have a high return means that people have to work a lot in period $t+1$. To get this, the interest rate for $t+1$ must be low......This is why we may get cycles.

Note that one can get a combination of these effects by introducing switches between the cases of gross substitutes and complements and by introducing some heterogeneity. A paper by Grandmont (Econometrica, 1985) discusses cycles in a model like this.

\[^7\text{This comes from evaluating (21) at } n^* \text{ using } U'(n^*) = g'(n^*).\]
5 Relation between exchange and production economies

We did all of the dynamic analysis for the production economy but we can also do it for an exchange economy.\textsuperscript{8} I don’t even have to work through all of the analysis – it should be enough to show you a connection in the consumer problem.

Suppose that $U(c_t, c_{t+1}) = U(c_t) + V(c_{t+1})$ where both $U(\cdot)$ and $V(\cdot)$ are increasing and concave. Further, suppose that when young agents have a positive endowment of the single consumption good, $e$, but no endowment when old. Let $s_t$ equal the real savings of a representative generation $t$ agent. Then agents solve max $U(e - s_t) + V(p_t p_{t+1})$.

The first order condition for this problem is $\rho_t V'(\rho_t s_t) = U'(e - s_t)$ where $\rho_t = \frac{p_t}{p_{t+1}}$. Analytically, this is really the same as the production model – look at the first order conditions but don’t get confused by the notation.

So we can go through the same exercises with respect to the relation between $s_t$ and $\rho_t$. The difference between gross substitutes and complements will be whether savings is an increasing or decreasing function of the interest rate. Also can redo the dynamics.

\textsuperscript{8}We did this for Shell’s economy but not for the more general model of Gale.