Notes on Money and Business Cycles

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1 Introduction

These notes are concerned with one answer to a fundamental question in macroeconomics: What causes the recurrent ups and downs we call the business cycle? For years, based largely on the work of Friedman-Schwartz (*A Monetary History of the United States*), the emphasis in macroeconomics was on monetary theories of economic fluctuations. The Great Depression in the United States was explained by the inadequate response by the monetary authorities to prevailing conditions. Earlier downturns in the economy could be traced to monetary instability, including bank runs. Furthermore, looking at the size of movements in output and employment, one needs to think of fairly large exogenous shocks which are able to produce these large movements. One natural place to look is at monetary aggregates.

But, this view has been challenged by theory and evidence. On the theory side there are two issues: (i) generating a demand for money and (ii) overcoming the neutrality of money. These themes will be addressed below.

As you may know, the basic Arrow-Debreu general equilibrium model has no demand for money. We can generate a demand for money through the overlapping generations structure. So we will stick with that modeling approach. The alternatives seem quite arbitrary ($ in utility functions, cash-in-advance constraints, etc.). The main drawback of the overlapping generations model is that it has difficulty sustaining a demand for money when there are other assets in the economy. That is, it is not a good model of the transactions role of money.

The second question on the effects of variations in the money supply is tough too. There is this wonderful thing called the "neutrality of money" which essentially states that the amount of money in the economy and the real decisions of individuals are totally independent. According to this view, variations in the money supply cause all prices in the economy to move together and so that no relative prices are altered. As a consequence, changes in the money supply have no real effects. This

\[1\text{There are many empirical papers trying to find evidence that money causes output. The results from these numerous studies are inconclusive, partly due to fundamental issues of identification of monetary shocks. Including these papers in these notes will take us too far afield.}\]
neutrality result is based on very strong classical thinking. So if we are to generate a monetary theory of the business cycle, we need to provide a reason why money is not neutral.

These reasons include:

- new money is distributed to agents in the economy in a way which disturbs the distribution of real wealth.
- changes in the money supply are not directly observable to agents in the economy.
- prices do not instantaneously respond to changes in the money supply.

We will see examples of the first point in these notes. This appears under the heading of non-proportional transfers. There is also an exercise which further highlights the redistributive effects of money creation.

The papers by Lucas and Azariadis that we will study stress the inability of agents to directly observe changes in the money supply. This is the key assumption in the vast literature called the new classical theories or the natural rate theories or the Lucas-Sargent-Wallace supply curve etc.

The papers on wage and price rigidities, on the reading list, work off of the inability of prices to respond to changes in the money supply. That approach builds upon the earlier contributions on fix price models by Barro-Grossman, Jacques Dreze, J-P Benassy, J-M Grandmont and Edmond Malinvaud.

To understand why one must go to so much trouble in order to understand the connection between money and economic activity, we will start with a restatement of the classical neutrality results. Also discuss this issue of distribution and then go into these other theories. Part of this will allow us to understand rational expectations equilibrium which is a modification of perfect foresight for uncertain environments.

## 2 Money neutrality: Observable Money Supply

We start with economies under the assumption that the current money supply is observable to agents. This will help us to understand why we analyze models in which money supply changes are assumed to be imperfectly observed. We do this with the overlapping generations model with production.\(^2\)

This discussion covers the same material as in previous class notes and will be skipped in lecture.

### 2.1 Overlapping Generations Model with Production

This is a review of this model from the earlier notes. Consider a simple overlapping generations models with production. There are \(N\) agents born each period who produce in youth and consume

\(^2\)As an exercise, try to reformulate this discussion using the overlapping generations model with saving.
in old age. Time is denoted by $t = 1, 2, 3, \ldots$. Agents born in generation $t$ have preferences defined over consumption when old ($c_{t+1}$) and work in youth ($n_t$). These agents are endowed with a unit of leisure time in youth, so that we require $0 \geq n \leq 1$ for all $t$.

Preferences are represented by $u(c_{t+1}) - g(n_t)$ with $u(\cdot)$ strictly increasing and concave and $g(\cdot)$ strictly increasing and strictly convex. The production technology is quite simple: output equals input. The per capita money holdings of the initial old is $M$.

For now, assume that $M_t = M$ for all $t$. Does the equilibrium values of real variables depend on $M$? If not, then this is one form of money neutrality. Further, if $M_t$ is some random process, we can see if the equilibrium values of the real variables depend either on the properties of this process or the realized values of the money supply. We will come to this later.

In working through this model, we will characterize the set of equilibrium paths for output, employment and consumption. If the set of equilibria is independent of $M$, then the model displays the classical dichotomy. I.e. the determination of real variables is independent of the nominal variables. When variations in the money supply have no real effects, we say that money is neutral. Does our economy satisfy these properties? To find out, we characterize the set of equilibria for our simple production economy.

### 2.2 Maximization problem of young agent

In youth, a representative generation $t$ agent solves

$$\max_n u\left(\frac{p_t n}{p_{t+1}}\right) - g(n)$$

where we have used the budget constraint of $c_{t+1} = \frac{p_t n}{p_{t+1}}$.

Note that the notation distinguishes this as a particular agent in the economy since we use $n$ as the generic choice variable. Note too that there is no explicit mention of money in this optimization problem. In fact, earnings in youth ($p_t n$) are held in the form of money to finance consumption in old age.

The solution of this optimization problem is

$$\rho_t u'(\rho_t n) = g'(n).$$

Here $\rho_t = \frac{p_t}{p_{t+1}}$ and is the real rate of return on working between periods $t$ and $t+1$. This is, the worker receives $p_t$ units of money per unit produced in period $t$ and can buy $\frac{1}{p_{t+1}}$ units of goods with each unit of money in period $t+1$. This condition is quite similar to that used in static models where $\rho_t$ plays the role of the real wage.
2.3 Market clearing

The condition for goods market clearing is that the total quantity of goods supplied equals the
amounted demand. The condition for money market equilibrium is that the money demanded by
the young equals the stock of money held by the old.

Since all agents are identical, let \( n_t \) be the amount supplied by an arbitrary generation \( t \) young
agent. Since the only purchases of goods comes from the old agents who each hold \( \overline{M} \) units of
fiat money, per capita real demand is \( \overline{M} / p_t \). Thus we can represent goods market clearing in period
\( t = 1, 2, 3, \ldots \) as \( n_t = \overline{M} / p_t \).

Alternatively, we can look at money supply equal to money demand as

\[
n_t p_t = \overline{M}. \tag{3}\]

As is clear, if the goods market clears, so does the market for money. This is a consequence of
Walras law.

2.4 Equilibrium

A perfect foresight equilibrium is a sequence of prices and employment choices in each period, \( p_t, n_t \)
for \( t = 1, 2, 3, \ldots \) such that market clearing and individual optimization hold for all \( t \).

Note that we think in terms of sequences of prices since the choice of a representative generation \( t \) individual
depends on \( t \) and \( t + 1 \) variables.

The easiest way to find equilibria for this model is to substitute the market clearing condition
into the optimization condition yielding:

\[
n_{t+1} u'(n_{t+1}) = n_t g'(n_t) \tag{4}\]

for \( t = 1, 2, 3, \ldots \). Letting \( U(c) = cu'(c) \) and \( G(n) = ng'(n) \), this condition becomes

\[
U(n_{t+1}) = G(n_t) \tag{5}\]

This is a difference equation in \( n_t \) which can be analyzed once the properties of the two func-
tions \( U(\cdot) \) and \( G(\cdot) \) are known. Any feasible \( n_t \) sequence which satisfies (5) is a perfect foresight
equilibrium. The equilibrium price sequence is then computed from the market clearing condition.

For now, we will be interested in the steady states of this model, i.e. the solutions such that
\( n_t = n^* \) for all \( t \). The non-autarkic steady state satisfies \( U(n^*) = G(n^*) \) or equivalently \( u'(n^*) =
g'(n^*) \).

\[\text{The earlier notes on the overlapping generations model with production provides a more complete derivation of these conditions.}\]

\[\text{The earlier notes discussed non-steady state solutions.}\]
If \( u(\cdot) \) is strictly concave and \( g(\cdot) \) is strictly convex, then there exists a unique steady state with 
\( n^* > 0 \).\(^5\) Further, in a steady state the real rate of interest is constant at \( \rho = 1 \).

So what happens when the initial money supply is varied? Note that the difference equation which characterizes the real (as opposed to nominal) variables of the system did not contain any \( M \) terms. Since the real allocations are independent of \( M \), prices must vary in proportion with this variable so that markets clear. This then implies that the ratio of prices, \( \rho_t \), is independent of \( M \). Since all real decisions are dependent on \( \rho_t \), this is another way to see the neutrality of money. Variations in \( M \) alter all prices proportionately, leaving unchanged and hence leaving real decisions unchanged. So the initial money supply has no effect on the equilibrium allocations in these models. What about variations of the money supply in some other period? We turn to that now.

## 3 Stochastic Proportional Transfers

We again consider the same overlapping generations production economy with one modification. Suppose that the money supply in period \( t \), denoted \( M_t \) is random and is determined from

\[
M_t = M_{t-1}x_t. \tag{6}
\]

Here \( x_t \) is a random variable with a cumulative distribution function of \( F(\cdot) \) and a pdf of \( f(\cdot) \). So each period the money supply is either augmented or reduced by \( x_t \). Assume that the random variable \( x \) is iid so that the pdf, \( f(\cdot) \), is independent of time and any of the past realization of \( x \).

Throughout we assume that agents in the economy know the functions \( F(\cdot) \) and \( f(\cdot) \). We are specific below about what they know about the realization of \( x \) when they make decisions.

When describing the process of money creation, it is critical to state how the new money comes into the economy. Each time someone tells you about the effects of changes in the money supply, make sure that you understand why money is held and how changes in the money supply are brought about. Examples of the latter are

- proportional transfers
- non-proportional transfers
- government purchases financed by printing $
- open market operations

As we will see with the Lucas (JET, 1972) model (and discussed further below), proportional transfers will be assumed. In this way, money non-neutralities are not caused by arbitrary redistributions of wealth. Let’s first look at their consequences and then return to the alternatives later.

\(^5\)This assumes the conditions outlined in the earlier notes for an interior solution to (5) hold. That is: \( u'(0) > g'(0) \) and \( u'(1) < g'(1) \).
So assume that the new money created is distributed in proportion to the money holdings of the agents. Also everyone knows that they will receive this transfer though the exact level may be random.

3.1 Individual Optimization

The appropriate budget constraint for a representative generation $t$ agent is

$$c_{t+1} = \frac{p_t n x_{t+1}}{p_{t+1}}$$

where $n$ is the level of employment by this agent and $p_t$ is in money price of goods in period $t$.

In (7), the money transfer is in proportion to the amount of money that the agent is holding after selling goods in youth ($p_t n$). This process of introducing new money is just like the government paying interest on money holdings.

With this budget constraint in mind, each young agent maximizes

$$\max_n E_{x_{t+1}, p_{t+1}, t} u\left(\frac{p_t n x_{t+1}}{p_{t+1}}\right) - g(n)$$

For any distribution function for $p_{t+1}$, the first order condition associated with this problem is

$$E_{x_{t+1}, p_{t+1}} U\left(\frac{p_t n x_{t+1}}{p_{t+1}}\right) = G(n)$$

where $U(c) = cu'(c)$ and $G(n) = ng'(n)$ as before.

3.2 Market Clearing

The condition for market clearing in period $t = 1, 2, 3...$ is simply the equality of money supply and money demand. Let $n_t$ be the average level of output and employment by the generation $t$ agents, then we can express market clearing in period $t$ as:

$$n_t p_t = M_t.$$  

This is close to (3) except that here the money supply changes over time as indicated by the subscript $t$.

3.3 Rational Expectations Equilibrium

What does the expectations operator refer to in (8)? Here the expectation is taken with respect to the unknown future variables: $x_{t+1}$ and $p_{t+1}$. Now what are the distributions for these variables? Well the distribution of $x_{t+1}$ is exogenous and is given by $f(\cdot)$. But the future price is endogenous, where does its distribution come from? This is where we enter into consideration of the concept of rational expectations equilibrium.
In general, we can think of the price in any period as dependent on the past history of monetary shocks as well as the current value of the money shock. Then, to determine an agent’s decision, the expectation is taken with respect to the distribution of future prices induced by this equilibrium relation. So the key to a rational expectations equilibrium is that we are forecasting the future values using all available information of the endogenous variables based upon the equilibrium relations between these variables and the exogenous variables. So a rational expectations equilibrium is defined through functions relating the endogenous to exogenous variables. These functions are used to generate individual behavior and then the individual behavior is consistent (i.e., markets clear and expectations are consistent) with these functions. Note then that this is an equilibrium concept and is more than requiring that people use all available information.

A stationary rational expectations equilibrium, hereafter SREE, is characterized by two function of the state of the system \((M, x)\). Here \(M\) is the stock of money inherited from the past and \(x\) is the current shock. Nothing else about the past history of the economy has any impact on the outcome in the current period.\(^6\)

One function in a SREE, \(n(M, x)\) is the employment rule and the other, \(p(M, x)\), is the price function. Note that time does not enter these functions: this is the essence of stationarity. In a stationary rational expectations equilibrium, these two functions are consistent with individual optimization and market clearing. In particular, the price function is used to form expectations about future prices given information today.

We will search for a rational expectations equilibrium in which the price level in any period depends on the money supply from the previous period and the money shock in that period. Let’s conjecture that a rational expectations equilibrium has a price function of the following form \(p = QMx\) where \(Q\) is a constant we will determine later. This is a good guess as we suspect that money will be neutral in this economy or, equivalently, prices will be proportional to the stock of money.

Suppose that individuals take this function as given and optimize. Substituting this conjectured function into the first order condition yields

\[
U(n_{t+1}) = G(n_t). \tag{11}
\]

and a stationary rational expectations equilibrium employment rule is simply \(n^*\) satisfying

\[
u'(n^*) = g'(n^*). \tag{12}
\]

From the strict concavity of \(u(\cdot)\) and the strict convexity of \(g(\cdot)\) there will exist a unique \(n^*\) satisfying (12).

\(^6\) Though, in some cases, history can matter if everyone believes it does even if the historical variables have no fundamental effect on preferences, technology or the information used to forecast future random variables.
Do markets clear with these decisions and this price function? With these functions, market clearing, given in (29), becomes
\[ QMx^* = Mx \] for all \((M, x))\). Thus \(Q = \frac{1}{\sigma} \). Since we know what \(n^*\) is we can immediately determine \(Q\).

So we conjectured a price function. Agents take this as given thus inducing real decisions. Then we check our consistency requirement – market clearing – and find that this is met. Hence we have a stationary rational expectations equilibrium.

In this equilibrium money is neutral. Variations in \(x\) are met by equal variations in the current price level. Even though agents cannot forecast the future values of \(x\) and \(p\), they do know that the ratio \(\frac{x}{p}\) is constant in this equilibrium. So they make decisions actually under certainty and money is completely neutral. Note that the real return on working is \(\frac{p_{t+1}x_{t+1}}{p_{t+1}}\) which equals one in this SREE equilibrium.

Two other comments. First there may be other equilibria in which money is not neutral. People can believe that money matters and so it does! Second, these results do depend on the proportional transfers and the fact that agents forecast these transfers. We will come back to this first point later and focus on the second one for now.

4 Non-proportional transfers

The point here is to consider a model with lump-sum money transfers. This serves to illustrate the inflation tax and also allows us to appreciate the role of proportional transfers. For this model, we retain the basic assumptions from the simple production model.

Assume that the money supply follows \(M_{t+1} = M_t(1 + \sigma)\). So here \(\sigma\) is the growth rate of the money supply. For simplicity, assume that it is non-stochastic.

When the new money is created it is distributed to agents as a lump sum. This is the difference between this model and the previous one. So the individual’s budget constraint is
\[ p_{t+1}c_{t+1} = p_tn_t + \gamma_{t+1} \] (14)

In this expression \(\gamma_{t+1}\) is a lump sum transfer and \(p_tn_t\) are money holdings from youth. This transfer is taken as given by the individual agent in solving the maximization problem. I.e., it is not viewed as interest paid on money holdings in contrast with the model of proportional transfers.

Of course, there is a link between the lump sum transfer and \(\sigma\) since all agents receive the same transfer. This is given by
\[ \gamma_{t+1} = \sigma M_t. \] (15)

As in the previous model, goods market clearing requires that
\[ p_tn_t = M_t \] (16)
for $t = 1, 2, 3, \ldots$.

Individual optimization implies

$$\frac{p_t}{p_{t+1}} u'(\frac{p_t n_t + \gamma_{t+1}}{p_t}) = g'(n_t).$$

(17)

The budget constraint, which links $c_{t+1}$ to the prices and employment levels, has been substituted into this expression. Note that the first term on the left, which is the real return to working, does not depend directly on the money transfer. Compare this to the first order condition for the proportional transfer case.

Substituting the market clearing conditions into this first order condition, (17) above, (just as in the other versions of this model) implies

$$(n_{t+1})u'(n_{t+1}) = (1 + \sigma)n_t g'(n_t)$$

(18)

$$U(n_{t+1}) = (1 + \sigma)G(n_t)$$

(19)

The steady state is $n^*(\sigma)$ satisfying

$$U(n^*) = (1 + \sigma)G(n^*)$$

(20)

or

$$u'(n^*) = (1 + \sigma)g'(n^*).$$

(21)

Note that $\sigma$ matters here! The steady state monetary equilibrium depends on $\sigma$ as do all of the equilibrium paths. Why is this? Increases in $\sigma$ affect the level of prices in the economy. But if I work a little more, all I receive is $p_t$ and the price level next period, $p_{t+1}$, depends on $\sigma$. Unlike the model of proportional transfers, the money infusion does not influence the marginal amount of nominal money balances I take into old age. The transfer does influence the price level so that it acts as a tax on my work – for higher $\sigma$, the marginal return on my work is lower. This is not true in the proportional transfer model. Another way to say it is that in equilibrium, the real rate of return depends negatively on $\sigma$. So increases in $\sigma$, lead to lower returns on working through the inflation tax. In the steady state, $\rho = 1/(1 + \sigma)$.

To see this more formally, we can perform simple comparative statics on the equation characterizing the steady state to show that $n^*$ is a decreasing function of $\sigma$. I.e., differentiating (21) implies

$$\frac{dn^*}{d\sigma} = \frac{g'(n^*)}{a''(n^*) - g''(n^*)\gamma(1 + \sigma)}$$

(22)
This is clearly negative. So the inflation tax reduces employment by taxing work effort. Note too that there is only a substitution effect here since the “tax revenues” from the inflation tax are redistributed in a lump sum manner.\footnote{As an exercise on this point, consider a static labor supply problem where productivity is $A$ and the government levies a labor tax at rate $\tau$. Suppose that all revenues collected by the government are transferred back to the households in a lump-sum manner. Show that the effect of $\tau$ on labor supply includes only a substitution effect.}

Note that this gives us a simple model of money non-neutrality. But it gives us the “wrong” correlations between money growth and output! I.e. here as $\sigma$ increases, people work less due to the inflation tax. So this is not consistent with observed correlations (i.e the Phillips curve) in the macroeconomy where inflation and economic activity were positively correlated. It might be important though over longer time spans and across countries.

## 5 Money non-neutralities and intertemporal substitution

This section provides a brief literature review leading to the models in the next section. You can take some time to read these papers at your leisure.

Friedman’s American Economic Association presidential address (\textit{American Economic Review}, March 1968) provides a discussion of the role of monetary policy. In this presentation, Friedman argues first that monetary does have real effects (i.e. money is not neutral) and that these real effects are not long term in nature. That is, monetary policy can only have short term real effects. Friedman introduces the notion of the natural rate of unemployment as the long run position of the economy. Deviations of unemployment from the natural rate are possible but these are only temporary. They can be brought about by changes in the money supply which are not anticipated. Friedman says that "it takes time for people to respond to a new state of demand" and actually talks about prices and wages being set for some time in the future on the basis of past experience. So this sounds like a sticky wage and price story. Soon expectations will adjust and the economy will return to the natural rate of unemployment. So Friedman concludes that "there is a temporary tradeoff between inflation and unemployment but no permanent one."

This article, along with the complementary work by Phelps, set the stage for the development of natural rate theory, intertemporal substitution theories and rational expectations models as a means of understanding the impact of monetary policy on the economy. So, money is not neutral here because changes in the money supply are not immediately observed by all agents. This “fooling” is temporary so that money is neutral in the long run.

What is left open is the nature of expectations. Friedman was not specific about this issue. Lucas and many others have been. The basic supply equation which captures much of this intuitive argument is

\begin{equation}
Y_t = Y_n + \gamma(p_t - E_{p_{t+1}}).
\end{equation}

(23)
In this equation, $Y_n$ is the natural rate of output and is the level of production that would arise if agents could perfectly determine the current state of the economy. This would be the equilibrium in a purely classical model. For simplicity this is not indexed by time as it might be if one allowed for economic growth. The parameter $\gamma$ is a constant at this stage though we will discuss its determinants later. The second term is the difference between the actual and the expected price levels. One "story" is that some agents (like workers) make decisions on labor supply and output based on knowing current prices $p_t$ and without knowing future prices, $E_{t+1}p_{t+1}$. So an increase in the difference between these two variables is like an increase in the expected real wage and thus may induce an increase in output and employment since $\gamma > 0$. So unanticipated increases in the price level can increase output.

An alternative approach, Lucas [American Economic Review 1973], focuses on the importance of a multi-sector economy in which agents know their own output prices but do not know the cost of consumption goods. (See Chapter 14 of Barro, Macroeconomics, too.) So workers must forecast the value of their nominal wage while firms can calculate the value of their real (product) wage easily.

To see this more formally, suppose that workers (as in the OG model) live for two periods, work in youth and consume in old age. Their labor supply then depends on the current wage they receive ($\omega_t$) and their expectation of the future price level, $E_{t+1}p_{t+1}$. Firms labor demand depends solely on the current real wage $\frac{\omega_t}{p_t}$. Consider a graph of the labor market with the nominal wage on the vertical axis. Now consider the construction of the aggregate supply curve. As we vary $p_t$, labor demand shifts around. What about labor supply? Well if workers expectations of future prices vary with $p_t$, then labor supply will shift with demand and we will have a vertical aggregate supply curve.

Alternatively, expectations of future prices could be less responsive to current prices. This could occur because expectations are slow to adjust (as in adaptive expectations) or because agents see that current prices are higher but don’t know if they are permanently or temporarily higher. This imperfect information on the source of price fluctuations provides a basis for the non-neutrality of money in an economy with rational expectations, optimizing behavior by all agents and market clearing. Variations in the money supply are not directly observable and agents interpret higher prices as, in part, a consequence of real shocks and so they respond. Can we write down this model? We will study the paper by Azariadis (AER, 1981) to see this.

The papers by Lucas (AER,1973) and Sargent-Wallace (JPE,1975) are good papers to look at to see the implications of an aggregate supply curve of this type. (Note though some of the differences in the timing of the variables). What they do is, in its simplest form, to combine this aggregate supply relationship with an aggregate demand curve from IS LM analysis. The monetary and fiscal policy variables, as well as the other parameters, can be stochastic. They compute the rational expectations equilibrium for this model and then derive a number of important propositions.

In general, a number of important propositions emerge from these models:
• the natural rate of output is independent of the money supply process.

• anticipated changes in the money supply have no real effects implying that the choice of policy rule is irrelevant

• the best policy is non-stochastic.

These properties are listed at the start of Azariadis' article as forming the basic implications of the natural rate theories. Sargent-Wallace view their paper as an example of the power of the rational expectations hypothesis but did not view their results as general. Others have – so it is important to see whether these statements stand the test of a rigorous model.

These are old papers. We study them for two reasons. First, Lucas (Journal of Economic Theory, 1972) represented a conceptual breakthrough in terms of modeling a rational expectations equilibrium. That contribution survives the initial point of the article. Second, the arguments presented in these papers are still used in thinking about money non-neutralities.

6 Imperfect Information and the Business Cycle

This section draws upon the papers by Lucas (1972) and Azariadis (1981) on the reading list. The presentation is close to Lucas but we have only a production decision. We then turn to the example explored in Azariadis.

6.1 Overview

The model we study builds from the overlapping generations model with production. We assume an overlapping generations model so that we have a demand for money. The money supply varies stochastically through proportional transfers. In addition, real shocks (variations in tastes, productive opportunities, population) are also present. Current models in the real business cycle literature, that we study later in the course, rely solely on these types of shocks. With some assumptions on what agents observe, this combination of real and nominal shocks creates confusion – this is the first principal element of the theory.

As they live two periods, individuals must forecast future prices when they choose how much to work. To do this, they need to extract from current variables the values of the nominal shocks. The money shocks are important because variations in the money supply are permanent – \( M_{t+1} = M_{t}x_{t+1} \). So if we know that the money supply is high today, we will forecast high prices for tomorrow and this shouldn’t influence real decisions – i.e. money will be neutral as we understood from the discussion in Section 3.

But, the real shocks will influence our decisions. High real shocks may imply that real returns are temporarily high so that agents should work more. So agents want to respond to real shocks –
this is the intertemporal substitution hypothesis and is the second critical element. But, because of confusion, agents cannot distinguish between real and nominal shocks and so are induced to intertemporally substitute when money shocks occur.

Hence the two key elements – confusion and intertemporal substitution– combine to create money non-neutralities. This happens even though new money is transferred proportionately to agents, markets clear and agents have rational expectations.

Lucas (1972) was the first paper to put this all together. This contribution adds real content to the stories of Friedman and Phelps.

The Lucas paper not only provides a micro-theoretic mechanism for relating real and nominal variables but is also the first example of a rational expectations equilibrium. Even if this model is not "in vogue" today, we still study it as it provides numerous insights into the building of macroeconomic models.

First look at implications for correlations and then at some policy issues. The Lucas (1972) paper was "simplified" to a linear structure which was then used for policy analysis (Sargent-Wallace, JPE, 1975). A variety of policy conclusions emerged which were not consistent with the original theoretical formulation. Hence, Azariadis (AER, 1981) is important as it points out that many results on policy derived for the simple models do not hold in the general equilibrium, micro theoretic setting.

6.2 Model

Here are the essential elements of the model.

Demographics

• generation of size 1 is born each period

• Two islands (sectors) in the economy. The population on island 1 is $\theta$ while the population on island 2 is $1 - \theta$. $\theta$ is a random variable which is iid.

• agents live for 2 periods, never leave their island

• $t = 1, 2, 3...$

Preferences

• work in youth, consume when old

• Generation $t$: $u(c_{t+1}) - g(n_t)$ with $u'(\cdot) > 0, u''(\cdot) < 0, g'(\cdot) > 0, g''(\cdot) > 0$

• same preferences for each generation

• Gross Substitutes: $u'(c) + cu''(c) > 0$

Endowments
• a single unit of leisure time in youth.

• Nothing in old age

Technology

• \( y_t = n_t \)

• no storage technology

Money

• initial aggregate money supply is \( 2M_1 \)

• split evenly across two islands: each island starts with \( M_1 \)

• proportional money shocks each period drive the money supply on each island, \( M_{t+1} = M_t x_{t+1} \)

Uncertainty and Information structure

• Real Shocks (\( \theta_t \)): Not observed by any young agents of generation \( t \)

• Money Shocks (\( x_t \)): Known by old but not observed by generation \( t \) young agents

• Information: Young agents observe the price level on their island: they do not observe the realizations of \( x \) and \( \theta \) directly. They also know the money supply from the previous period.

The information structure is the key to the model. Young agents are born to a particular island and thus do not have information about the aggregate state of the economy. They see a price on this island (their market) and use this price to infer the return to working. This return to working depends on the price next period. What inferences about the price next period should they draw from observations on the price this period? The answer depends on what caused a price variation on a particular island today.

If I am on island 1, a low \( \theta \) today means that there are relatively few young people on my island. This tells me nothing about the number of young people on this island tomorrow. We term this a real shock because it influences one of the primitive elements of the economy – the distribution of traders. Agents will generally want to respond to the price changes induced by the real shocks.

If I am on island 1, a high \( x \) today means the old people have lots of money. But a money shock is usually neutral: I expect prices to be higher next period as well and thus I do not want to vary my labor supply.

But in this economy there is the potential for confusion between the real and nominal shocks. Agents do not observe \( x \) and \( \theta \) independently. Rather, they observe that \( p_t \) varies and then they must decide on the source of these fluctuations. They care because variations in the money supply have more permanent effects than do variations in the population. I.e., we will see that \( p_{t+1} \) depends on \( x_t \) through \( M_t \) but is not dependent on \( \theta_t \).
6.3 Competitive Equilibrium

For this discussion, we confine attention to one of the islands, the other will be determined symmetrically. We first look at the optimization problem of an agent and then turn to market clearing.

6.3.1 Optimization

The representative generation $t$ agent on island 1 faces the following budget constraint:

$$p_{t+1}c_{t+1} = p_t n_t x_{t+1}$$  \hspace{1cm} (24)

where $p_t$ is the money price of goods in period $t$. So the right side has the nominal income earned in youth, $p_t n_t$, times the stochastic money transfer in old age, $x_{t+1}$. The left side is nominal spending on goods in period $t + 1$.

Substituting this into their objective function, agents maximize

$$E(p_{t+1}, x_{t+1} | p_t) \{ u(p_t n_t x_{t+1} / p_{t+1}) \} - g(n_t)$$  \hspace{1cm} (25)

where the components of utility were described earlier.

The key to this optimization problem is that expectations are taken with respect to the two variables that are not known to agents at the time they decide how much to work: $x_{t+1}, p_{t+1}$. This expectation is taken conditional on the observed current price, $p_t$. So the current price not only influences the budget constraint but also provides some information on the current values of the shocks, $x_t$ and $\theta_t$, which influences agents’ forecasts of $p_{t+1}$.

The interior solution to this maximization problem is

$$E(p_{t+1}, x_{t+1} | p_t) \{ (p_t x_{t+1} / p_{t+1}) u'(p_t n_t x_{t+1} / p_{t+1}) \} = g'(n_t).$$  \hspace{1cm} (26)

The left side is the expected marginal gain from working a bit more. This is the real wage, $(p_t x_{t+1} / p_{t+1})$, times the marginal utility of future consumption, $u'(p_t n_t x_{t+1} / p_{t+1})$. The right side is the known marginal cost of working more.

6.3.2 Market Clearing

Money market clearing on island 1 requires that the demand for money equal the supply of money on a per capita basis.

$$p_t n_t \theta_t = M_t = M_{t-1} x_t$$  \hspace{1cm} (27)

From this expression, the price level in period $t + 1$ will depend on the shocks in that period, $x_{t+1}$ and $\theta_{t+1}$, and the money supply that was inherited from the past through the money holdings of the old, $M_t$. This is where the permanent effect of $x_t$ relative to $\theta_t$ comes into play. Variations
in \( x_t \) influence \( p_{t+1} \) through \( M_t \) so that agents who are young in period \( t \) want to disentangle the nominal and real shocks in that period given that they cannot observe \( M_t \) directly.

A similar market clearing condition holds on the other island where the fraction of the young is just one minus the fraction of the young coming to island 1. With the budget constraint holding, if the money market clears, so will the goods market.

### 6.3.3 Stationary Rational Expectations Equilibrium

Confine attention to a stationary rational expectations equilibrium. By stationary, we mean that none of the relationships determined in equilibrium will vary by the passage of time. By rational expectations equilibrium we mean that the expectations used to determine the amount of work by young people are not arbitrary. To decide on how much to work, these agents need to forecast both \( x_{t+1} \) and \( p_{t+1} \). The distribution of \( x_{t+1} \) is known to them but the distribution of \( p_{t+1} \) is determined in equilibrium.

Lucas deserves enormous credit for characterizing a rational expectations equilibrium for this economy. To do so, we need to find a price function such that the price in period \( t \) is dependent on the realizations of the random variables occurring in period \( t \) as well as the history of these variables. For simplicity, Lucas confined attention (he made a guess and verified that it worked) to a stationary price function which makes period \( t \) prices dependent on \( \theta_t, x_t \), and \( M_{t-1} \). From this function, one can then calculate the conditional distribution of \( p_{t+1} \) since future prices depend on the future values of \( x \) and and the inherited money supply, \( M_t \).

So we call \( p(x, \theta, M) \) an equilibrium price function if the actions of agents, taking this price function as given, reproduce it from the market clearing conditions. That is, take a candidate function, use it to determine agents budget constraint and forecasts of future prices. If the actions that they then take satisfy the market clearing restrictions at these prices, then we have an equilibrium. This is similar to the notion of equilibrium we employed in the case of stochastic money transfers although this is a bit more complicated because of the presence of the conditional expectation.

So the equilibrium price in period \( t \) is given by \( p_t = p(x_t, \theta_t, M_{t-1}) \). This makes clear that in the price function, both \( x \) and \( \theta \) refer to the current period and \( M \) is the inherited money supply.

To prove the existence of a rational expectations equilibrium is potentially difficult. It requires the use of a fixed point argument in function space. Lucas discusses this in the appendix of his paper. In general, proving existence in rational expectations models is made more difficult because of the extra role of prices – they provide information in making forecasts. So, the budget set may be continuous in prices but beliefs about the future may not be – non-existence may result. In fact, there may be multiple equilibria as well.

The most direct way to proceed is to conjecture a price function and then see if we can get it to work. Lucas does this by conjecturing that the prices must be proportional to the known inherited
money supply and dependent only on the ratio of \( \frac{x}{\theta} \). The first element of the conjecture goes back to the neutrality of money discussion. If the inherited money is observable to all agents, then its level cannot matter for real decisions. So period \( t \) prices must be proportional to \( M_{t-1} \). The second part of the conjectured price function reflects the fact that the ratio of \( \frac{x}{\theta} \) appears in the market clearing condition – so this is what agents can get from observing prices.

Conjectured price function is

\[
p_t = p(x_t, \theta_t, M_{t-1}) = M_{t-1} \phi(\frac{x_t}{\theta_t}) = M_{t-1} \phi(z_t)
\]

where \( z_t \equiv \frac{x_t}{\theta_t} \). So the unknown here is the stationary function, \( \phi(z_t) \). It is stationary in that the function itself does not change over time though the argument does.

Using this price function, market clearing implies that

\[
y_t = n_t = \frac{z_t}{\phi(z_t)} = \psi(z_t).
\]

We will use \( \psi(z_t) \) as the relationship between output and \( z_t \) in this economy.

Recall that young generation \( t \) agents observe the price on their island and are assumed to know the money supply from the previous period, \( M_{t-1} \). So if the conjectured price function holds, then knowing \( p_t \) agents know \( \phi(z_t) \). But do they know \( z_t \)? The following result from Lucas (1972) argues that \( \phi(z_t) \) reveals \( z_t \).

Lemma: \( \phi(z) \) is monotone in \( z \).

Proof: Assume not so that there exist some \( z_1 \neq z_2 \) with \( \phi(z_1) = \phi(z_2) \) for \( i = 1, 2 \).\(^8\) If \( \phi(z_1) = \phi(z_2) \), then output per person is the same since agents make decisions conditioning on \( z \). But \( \phi(z_1) = \phi(z_2) \) with \( z_1 \neq z_2 \) is not consistent with market clearing, (29).

6.3.4 Stationary Output Function

Our ultimate goal is to understand the function \( \psi(z) \) in a stationary rational expectations equilibrium. To do so, we follow steps from our earlier analysis of the overlapping generations model with production and substitute the market clearing conditions into individual optimization to obtain an expression in quantities alone.

The first step is to write the return on working in terms of the price function:

\[
\frac{p_t x_{t+1}}{p_t y_{t+1}} = \frac{M_{t-1} \phi(z_t) x_{t+1}}{M_t \phi(z_{t+1})} = \frac{\phi(z_t) x_{t+1}}{\phi(z_{t+1}) x_t}.
\]

Using the condition from market clearing of \( \psi(z_t) = \frac{z_t}{\phi(z_t)} \), we can write this as:

\[
\frac{p_t x_{t+1}}{p_t y_{t+1}} = \frac{\psi(z_{t+1}) \theta_{t+1}}{\psi(z_t) \theta_t}.
\]

\(^8\)Here the subscript refers to two values of \( z \) not time.
Since, in equilibrium $n_t = \psi(z_t)$, we can substitute this into the generation $t$ representative agent’s first-order condition, (26) to obtain:

$$E_{(\theta_t, \theta_{t+1}, x_{t+1}|z_t)}\left\{ \frac{\psi(z_{t+1})\theta_{t+1}}{\psi(z_t)\theta_t} \right\} u' \left( \frac{\psi(z_{t+1})\theta_{t+1}}{\theta_t} \right) = g'(\psi(z_t)).$$

(32)

Here we condition on $z_t$ since this is revealed to the agent. The expectation is then over the future random variables, $(x_{t+1}, \theta_{t+1})$, as well as the current unknown random variable, $\theta_t$. In this expression, the only unknown is the function $\psi(z_t)$.

The last step is to get rid of the notation for the time period. This is a stationary equilibrium and thus time, per se, is not relevant. We denote current value of some variable $q$ as $q$, its future value as $q'$ and its past value as $q_{-1}$.

$$E_{(\theta, \theta', x')}|z)\left\{ \frac{\psi(z')\theta'}{\theta} \right\} u' \left( \frac{\psi(z')\theta'}{\theta} \right) = g'(\psi(z))\psi(z).$$

(33)

A stationary rational expectations equilibrium is then a function, $\psi(z)$, which solves (33), for all $z$. The only unknown in (33) is the function $\psi(z)$. The future value of the random variables $(\theta', x')$ are integrated out using the distributions of $\theta$ and $x$. The expectation of $\theta$ conditional on $z$ is the part of the expectation which is determined in equilibrium.

Once we solve for the function $\psi(z)$ which solves (33) for all $z$, then we can find prices using the relationship $\phi(z) = \psi(x) = Qx$ where $Q$ is some constant. Further, $\psi(z) = \psi(x) = \frac{x}{\phi(z)} = \frac{1}{Q}$. Hence output is constant: $\psi(x) = \tilde{\psi}$ for all $x$. Once $\tilde{\psi}$ is determined, then we can solve for $Q$ and thus we know the price function as well.

The level of output and employment comes from solving (33) in this case. We find

$$u'(\tilde{\psi})) = g'(\tilde{\psi}).$$

(34)

This is the same equation we have seen before. The level of employment solving this equation is the same as the level of employment in the planner’s problem. This is also the same level of output and employment as in the stationary competitive equilibrium with valued fiat money discussed in section 2.
Case 2: Assume $x = 1$ with probability one and $\theta$ is random so $z = \frac{1}{\theta}$. Thus there is no confusion in this economy: prices reveal $z$ and this reveals $\theta$.

The expression for equilibrium is

$$E \theta' \{ \psi \left( \frac{1}{\theta} \right) \psi \left( \frac{1}{\theta} \right) \} = g' \left( \frac{1}{\theta} \right) \psi \left( \frac{1}{\theta} \right)$$

(35)

for all $\theta$.

From this, we will argue that as $\theta$ increases, $\psi \left( \frac{1}{\theta} \right)$ will fall. In words, if there are more young agents on your island, then $z = \frac{1}{\theta}$ will be lower and you will choose to work less.

To prove this, as $\theta$ increases, the left side of (35) falls. This follows from the assumption of gross substitutes: $cu'(c)$ is increasing in $c$. In order then for (35) to hold, $z = \frac{1}{\theta}$ must fall since $g'(n)n$ is increasing in $n$.

What about prices? Well from market clearing, the effect of $\theta$ on prices depends on how the product of $\theta \times n$ varies with $\theta$. Since agents make decisions based on their expected returns from working, in order for them to work less when increases, it must be the case that $p_t$ falls when $\theta_t$ increases. So the key here is that when we have more agents on your island, the perceived returns to working are lower since the price of output on our island is lower. This is because $\theta_t$ affects $p_t$ but not $p_{t+1}$ as shocks are independent over time.

This is where the intertemporal substitution hypothesis comes in. As $p_t$ varies, this induces agents to alter their labor supply decision by substituting leisure today for consumption tomorrow. That is, if $p_t$ rises, I am willing to substitute consumption tomorrow for leisure today. The strength of this intertemporal substitution effect is thus the key to whether this theory can explain business cycle behavior.

6.5 Both Shocks

Now that we have seen how the special cases operate, we can return to the economy of interest with both real ($\theta$) and nominal ($x$) shocks. From the two extreme examples, we see that agents wish to respond to variations in their real shocks. The problem is that agents do not observe $\theta$ and $x$ independently but instead observe $z$ which depends on these random variables.

There are two types of equilibria to consider. In the first, termed revealing equilibria, observations on $p_t$ are informative about both $x_t$ and $\theta_t$. We will discuss an example of this in Azariadis' paper). Here agents will respond to changes in $p$ induced by variations in $\theta$ but will not vary employment due to changes in the money supply, $x$. Thus money is neutral.

In the second type of equilibria, termed non-revealing equilibria, observations on $p_t$ are not fully informative about realizations of $x_t$ and $\theta_t$. This is the interesting case of confusion. It is here that variations in $x$ can have real effects since agents attribute some the resulting variations in $p$ to variations in which they wish to respond to!
Thus we focus on equilibrium with non-revealing prices. In this case, the equilibrium level of output is given by (33) which we write as

\[
\int \left[ E(\theta', x') \left\{ \frac{\psi(z')}{\theta} \cdot u'\left(\frac{\psi(z')}{\theta}\right) \right\} \right] dF(\theta|z) = g'(\psi(z))\psi(z) \tag{36}
\]

for all \( z \).

In writing the equilibrium condition in this way, we make clear that \( z \) matters only through its influence on the conditional distribution of \( \theta \). In words, agents see prices and from them know \( z \). Given \( z \) and some elementary statistics, they can compute the distribution of \( \theta \) given \( z \). This is used by agents to make decisions and thus appears in the equilibrium condition, (36).

Now the whole question of the influence of money depends on how agents modify the distribution of \( \theta \) in response to variations in \( z \). Lucas, and we follow, assumes, for all \( \hat{\theta} \), that the probability \( \theta \leq \hat{\theta} \) increases in \( z \). Thus, high \( z \) speaks for low \( \theta \) – observing a high value of \( z \) and this is at least partially attributable to low values of \( \theta \).

With this assumption, as \( z \) increases, we place more weight on low values of \( \theta \). This, combined with the assumption of gross substitutes implies that the left side of (36) increases as \( z \) increases.

Thus to ensure that (36) holds for all \( z \), \( \psi(z) \) must be an increasing function. In summary, if: (i) leisure and consumption are gross substitutes and (ii) high \( z \) speaks for low \( \theta \), then output is an increasing function of \( z \).

This is where the real effects of money emerge. High \( x \) implies that \( z \) goes up (on both islands) so that agents are tricked into working more. They are tricked because they believe the high \( z \) may reflect a decrease in \( \theta \). Hence we get a positive correlation between the money supply and the level of aggregate economic activity.

### 6.5.1 Examples: Azariadis

The paper by Azariadis assumes a particular utility function: \( c_{t+1} - (1/2)n_t^2 \). The first order condition for the optimization problem of a generation \( t \) agent is:

\[
n_t = E\left( \frac{p_t x_{t+1}}{p_{t+1}} | p_t \right). \tag{37}
\]

So variations in employment depend on how the perceived terms of trade vary with \( p_t \). In the notation used above, we want an equilibrium output function \( \psi(z) \) such that

\[
E(\theta'\psi(z'))E\left( \frac{1}{\theta} | z \right) = [\psi(z)]^2 \tag{38}
\]

for all \( z \). Any \( \psi(z) \) function solving this condition is an equilibrium. Since the future values of and \( z \) are independent of those today, we can split the expectation as above.
Azariadis defines $[M(z)]^2 \equiv E(\frac{\theta}{\theta} | z)$ and $k^2 = E(\theta \psi(z))$ where $k$ is a constant to be determined in equilibrium. Using these definitions, the equilibrium output function is:

$$kM(z) = \psi(z)$$

for all $z$. From this, we can find $k = E\theta M(z)$. Equation (39) gives us the desired positive relationship between $y$ and $z$ as $M(z)$ is an increasing function of $z$ by assumption.

Azariadis explores four special examples to focus on the natural rate theories he discusses in his introduction. For this discussion, see the article. I will briefly discuss the examples. We have already talked about his examples 1 and 2. I will focus on the case of noisy prices.

Assume $\theta$ takes on two values, $\theta_1 < \theta_2$. Assume $x$ takes on the same two values and that all probabilities equal 0.5. Then $z$ takes on three values with the associated probabilities. This is shown in Table 1.

Note that $M(z)$ is an increasing function of $z$ since $\theta_1 < \theta_2$. From (39), $\psi(z)$ is thus an increasing function. In this economy, money is not neutral.

Now let’s evaluate a proposition about the effects of anticipated monetary policy in this economy. What happens if the monetary authority announces to everyone that there is a change in the distribution of $x$. In particular, suppose that the monetary authority adds a constant $\lambda$ to the two possible realizations of $x$. In this case, there will generally be four possible values for $z$. Each of them will uniquely identify both $x$ and $\theta$.

This illustrates that known changes in the distribution of the money shock can have real effects. This happens because the inferences agents draw from prices will depend on the distribution of $x$.

When are money changes neutral? Suppose that we replace $x$ with $\lambda x$ for $\lambda > 0$. Then, and we can return to the Lucas model to do this, it is easy to see that the equilibrium must be independent of $\lambda$. I.e. conjecture that prices are proportional to $\lambda$, $p_t = \lambda M_{t-1} \phi(z_t)$. For this to be consistent with market clearing, output in period $t$ must be independent of $\lambda$.

To see that in fact output is independent of $\lambda$, you can resolve the model to obtain a version of (36). This condition will be independent of $\lambda$. Basically, all that has happened is that an additional part of the money supply in period $t + 1$ has become predictable. Instead of knowing that $M_t$ will equal $M_{t-1}$ times some noise, we know that $M_t$ equals $\lambda M_{t-1}$ times some noise.

<table>
<thead>
<tr>
<th>$z$</th>
<th>Probability</th>
<th>$[M(z)]^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>0.25</td>
<td>$\frac{1}{\theta_2}$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.25</td>
<td>$0.5(\frac{1}{\theta_2} + \frac{1}{\theta_1})$</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>$\frac{1}{\theta_1}$</td>
</tr>
</tbody>
</table>

Table 1: Example

---

9 We still assume that $x$ and $\theta$ are independent.