Solving Complex PDE Systems for Pricing American Options with Regime-Switching by Efficient Exponential Time Differencing Schemes

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In this paper we study the numerical solutions of a class of complex PDE systems with free boundary conditions. This problem arises naturally in pricing American options with regime-switching, which adds significant complexity in the PDE systems due to regime coupling. Developing efficient numerical schemes will have important applications in computational finance. We propose a new method to solve the PDE systems by using a penalty method approach and an exponential time differencing scheme. First the penalty method approach is applied to convert the free boundary value PDE system to a system of PDEs over a fixed rectangular region for the time and spatial variables. Then a new exponential time differencing Crank-Nicolson (ETD-CN) method is employed to solve the resulting PDE system. This ETD-CN scheme is shown to be second order convergent. We establish an upper bound condition for the time step size and prove that this ETD-CN scheme satisfies a discrete version of the positivity constraint for American option values. The ETD-CN scheme is compared numerically with a linearly implicit penalty method scheme and with a tree method. Numerical results are reported **to** illustrate the convergence of the new scheme.

Keywords: Numerical PDE, free boundary value problem, exponential time differencing, penalty method, American option, regime-switching.

1 INTRODUCTION

The penalty method approach has been employed in the literature for solving the American option pricing problem, see for example [8, 9, 13, 14, 15, 16, 20, 21, 25]. It is well-known that the American option pricing problem leads to a free boundary value problem to which a general closed-form solution does not exist. Initially proposed by Zvan et al. in [25], the fundamental idea of the penalty method approach is to remove the free and moving boundary from the American option pricing equation by introducing a properly chosen penalty term, resulting in a partial differential equation on a fixed rectangular region for the temporal and spatial variables. Many efforts have been devoted to developing fast numerical schemes for implementing the penalty method approach. One scheme is to apply the θ-method that includes the forward Euler, the backward Euler, and the Crank-Nicolson methods as special cases. However, as noted in [15, 16], a straightforward implementation of the θ-method would result in a nonlinear equation that requires a time-consuming iterative procedure at each time step. To avoid such complications, [15, 16] treat the nonlinear penalty term explicitly and propose a linearly implicit scheme that can solve the discretized pricing PDE quickly.

Along another line, regime-switching models have drawn considerable attention in recent decades in financial mathematics and computational finance, due to their capability of modeling non-constant and perhaps random market parameters (e.g. volatility, interest rate) and their comparably inexpensive computation. In this setup, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents randomly changing economical factors. Model parameters (drift and volatility coefficients) are assumed to depend on the Markov chain and are allowed to take different values in different regimes. As a result, both continuous dynamics and discrete events are present in the regime-switching models. Option pricing in regime-switching models has been a particularly active area of research, see for example [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 18, 19, 22]. In particular, Khaliq and Liu [14] extended the linearly implicit scheme in [15, 16] to American option pricing in the regime-switching geometric Brownian motion (GBM) model. However, this scheme is only first-order accurate in time and space. For a systematic treatment of both theories and applications of the regime-switching models, we refer the reader to the recent book by Yin and Zhu [24].

In this work we present and test a new scheme for the numerical solution of the American option pricing problem in the regime-switching model. This scheme combines the penalty method approach with an exponential time differencing Crank-Nicolson (ETD-CN) method, resulting in a fast numerical method. Note that with regime-switching, American option prices satisfy a system of m free boundary value problems, where m is the number of regimes considered in the model. An (optimal) early exercise boundary is associated with each regime. To solve the problem, we first apply the penalty method approach to convert the free boundary value system to an approximation system of partial differential equations (PDEs) over a fixed rectangular region for the temporal and spatial variables. Then, we employ an exponential time differencing Crank-Nicolson (ETD-CN) method to solve the approximation PDE system. This ETD-CN scheme utilizes an exponential time differencing Runge-Kutta approach followed by a $(1,1)$-diagonal Padé approximation of matrix exponential functions, and is shown to be second order convergent. Thus, it provides an efficient implementation of the penalty method approach for pricing American options with regime-switching. We establish an upper bound condition for the time step size and prove that under this condition the ETD-CN scheme satisfies a discrete version of the positivity constraint for American option values. **We note that the stability of the ETD-CN scheme has been discussed for European options in Kleefeld et al. [17].** We use two numerical examples, one with two regimes and another with four regimes to test the new ETD-CN scheme. We compare the ETD-CN scheme with the linearly implicit penalty method scheme **derived** in [14] and with the tree method **derived** in [18]. Numerical results are reported in the paper **to** illustrate the second order convergence of the ETD-CN scheme.

The paper is organized as follows. The American option pricing problem in the regime-switching model is introduced in Section 2. The penalty method approach is employed to obtain an approximation system of PDEs over a fixed rectangular region for the temporal and spatial variables. The new exponential time differencing Crank-Nicolson (ETD-CN) scheme is presented in Section 3. Section 4 is devoted to the positivity constraint for American option values. An upper bound condition for the time step size is obtained and it is shown that under this condition the positivity constraint is preserved by the ETD-CN scheme. Numerical experiments are carried out in Section 5. Numerical results obtained using three different schemes are reported and compared. Section 6 provides further remarks and concludes the paper.

2 THE PDE SYSTEMS AND THE AMERICAN OPTION PRICING IN REGIME-SWITCHING MODEL

We consider a continuous-time Markov chain α_t taking values among m different states, where m is the total number of states (also known as regimes) considered in the economy. Each state represents a particular regime and is labeled by an integer i between 1 and m. Hence , the state space of α_t is given by $\mathcal{M} := \{1, \ldots, m\}$. Let the matrix $Q = (q_{ij})_{m\times m}$ denote the generator of α_t. In this paper we assume that Q is given. From Markov chain theory (see for example, Yin and Zhang [23]), the entries q_{ij} in Q satisfy: (I) $q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1, \ldots, m$.

Note that introducing a Markov chain α_t into the option pricing model will result in an incomplete market, implying that the risk-neutral measure is not unique. One can employ a regime-switching random Esscher transform to determine a risk-neutral measure for option pricing. See Elliott, Chan and Siu [6] for details. In what follows, we assume that the risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ is given. Let $\widetilde{B}_t$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \widetilde{\mathcal{P}})$ and assume it is independent of the Markov chain α_t. We consider the following regime-switching geometric Brownian motion (GBM) for the risk-neutral process of the underlying asset price S_t:

$$\frac{dS_t}{S_t} = r_{\alpha_t} dt + \sigma_{\alpha_t} d\widetilde{B}_t, \quad t \geq 0, \tag{2.1}$$

where σ_{α_t} is the volatility of the asset S_t and r_{α_t} is the risk-free interest rate. Note that both σ_{α_t} and r_{α_t} are assumed to depend on the Markov chain α_t, indicating that they can take different values in different regimes.

We consider an American put option written on the asset S_t with strike price K and maturity $T < \infty$. Let $V_i(S,t)$ denote the option value functions, where t denotes the time-to-maturity, the asset price $S_t = S$ and the regime $\alpha_t = i$. Then $V_i(S,t)$, $i = 1,\dots,m$, satisfy the following free boundary value problem:

$$\begin{cases} \dfrac{\partial V_i}{\partial t} - \dfrac{1}{2}\sigma_i^2 S^2 \dfrac{\partial^2 V_i}{\partial S^2} - r_i S \dfrac{\partial V_i}{\partial S} + (r_i - q_{ii})V_i - \displaystyle\sum_{l \neq i} q_{il} V_l = 0, \quad \text{if } S > \bar{S}_i(t), \\ V_i(S,t) = K - S, \quad \text{if } 0 \le S \le \bar{S}_i(t), \\ V_i(S,0) = \max\{K - S, 0\}, \\ \displaystyle\lim_{S \uparrow \infty} V_i(S,t) = 0, \\ \displaystyle\lim_{S \downarrow \bar{S}_i(t)} V_i(S,t) = K - \bar{S}_i(t), \\ \displaystyle\lim_{S \downarrow \bar{S}_i(t)} \dfrac{\partial V_i(S,t)}{\partial S} = -1, \\ \bar{S}_i(0) = K, \end{cases} \tag{2.2}$$

where $\bar{S}_i(t)$, $i = 1,\dots,m$ denote the (unknown) free moving exercise boundaries of the option.

As a first step, we extend the penalty method approach [20, 25] to the regime-switching American option pricing problem (2.2) as in Khaliq and Liu [14]. Let $0 < \varepsilon << 1$ be a small regularization parameter and $C > 0$ be a fixed constant. By adding the penalty terms [14]

$$\frac{\varepsilon C}{V_i^{\varepsilon}(S,t) + \varepsilon - q(S)}, \; i = 1,\dots,m, \tag{2.3}$$

respectively to the m PDEs in (2.2), where

$$q(S) = K - S, \tag{2.4}$$

we obtain the following approximation system of nonlinear partial differential equations on a fixed domain for (S,t):

$$\begin{cases} \dfrac{\partial V_i^\varepsilon}{\partial t} + A_i V_i^\varepsilon = \displaystyle\sum_{l \neq i} q_{il} V_l^\varepsilon + \frac{\varepsilon C}{V_i^\varepsilon + \varepsilon - q(S)}, \quad (S,t) \in [0, S_\infty] \times [0,T), \\ V_i^\varepsilon(S,0) = \max\{K - S, 0\}, \\ V_i^\varepsilon(0,t) = K, \\ V_i^\varepsilon(S_\infty, t) = 0, \end{cases} \tag{2.5}$$

where $V_i^\varepsilon(S,t)$, $i = 1, \ldots, m$ denote the solution of (2.5) which approximate $V_i(S,t)$, $i = 1, \ldots, m$, S_∞ is a sufficiently large number chosen as the upper bound for the asset price. The differential operators A_i, $i = 1, \ldots, m$ are defined by

$$A_i = -\frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2}{\partial S^2} - r_i S \frac{\partial}{\partial S} + (r_i - q_{ii}). \tag{2.6}$$

Next, we introduce m functions F_i, $i = 1, \ldots, m$ by

$$F_i(V_1^\varepsilon, V_2^\varepsilon, \ldots, V_m^\varepsilon) = \sum_{l \neq i} q_{il} V_l^\varepsilon + \frac{\varepsilon C}{V_i^\varepsilon + \varepsilon - q(S)}, \quad i = 1, \ldots, m. \tag{2.7}$$

We use step size k to discretize the time variable t. Let $0 < k \leq k_0$ for given constant k_0, and let $t_n = nk$, $0 \leq n \leq N$. Let $u_i(t_n) := V_i^\varepsilon(S, t_n)$, $i = 1, \ldots, m$. Then using a similar argument as in Kleefeld et al. [17, Section II], we can show that $u_i(t_n)$, $1 \leq i \leq m$, $0 < n \leq N$ satisfy the following recurrent formula:

$$u_i(t_{n+1}) = e^{-kA_i} u_i(t_n) + \int_0^k e^{-A_i(k-\tau)} F_i(u_1(t_n+\tau), u_2(t_n+\tau), \ldots, u_m(t_n+\tau))\, d\tau. \tag{2.8}$$

This equation (2.8) will be the basis for the exponential time differencing Crank-Nicolson (ETD-CN) scheme which we will present in the next section.

3 THE EXPONENTIAL TIME DIFFERENCING CRANK-NICOLSON (ETD-CN) SCHEME

Exponential time differencing (ETD) is a class of numerical schemes for approximating the integral part in solution representations like (2.8). Kleefeld et al. [17] consider a nonlinear parabolic initial-boundary value problem and derive a fully discrete second order ETD scheme. In this section we treat a different problem, namely, the American option pricing problem in the regime-switching model and derive the ETD-CN scheme for the approximation solution of the system (2.8).

Consider $t \in [t_n, t_{n+1}]$. Let $F_{i,n} := F_i(u_1(t_n), u_2(t_n), \ldots, u_m(t_n))$, i.e., the function F_i evaluated at the left endpoint t_n. By setting F_i equal to the constant $F_{i,n}$ for $t \in [t_n, t_{n+1}]$ in (2.8), we have

$$a_i(t_n) := e^{-kA_i} u_i(t_n) + \int_0^k e^{-A_i(k-\tau)} F_{i,n}\, d\tau = e^{-kA_i} u_i(t_n) - A_i^{-1}(e^{-kA_i} - I) F_{i,n}, \tag{3.1}$$

for $i = 1, \ldots, m$. We will use $a_i(t_n)$ as an intermediate prediction for $u_i(t_{n+1})$. Next we approximate the functions F_i in the interval $t \in [t_n, t_{n+1}]$ by

$$F_i(u_1(t), \ldots, u_m(t)) \approx F_{i,n} + (t - t_n) \frac{F_i(a_1(t_n), \ldots, a_m(t_n)) - F_{i,n}}{k}, \quad t \in [t_n, t_{n+1}]. \tag{3.2}$$

Using (3.2) in (2.8), we obtain,

$$\begin{aligned} u_i(t_{n+1}) &\approx e^{-kA_i} u_i(t_n) + \int_0^k e^{-A_i(k-\tau)} \left(F_{i,n} + \tau \frac{F_i(a_1(t_n), \ldots, a_m(t_n)) - F_{i,n}}{k} \right) d\tau \\ &= a_i(t_n) + \frac{F_i(a_1(t_n), \ldots, a_m(t_n)) - F_{i,n}}{k} \int_0^k e^{-A_i(k-\tau)} \tau\, d\tau \\ &= a_i(t_n) + \frac{1}{k} A_i^2 (e^{-kA_i} - I + kA_i)[F_i(a_1(t_n), \ldots, a_m(t_n)) - F_{i,n}]. \end{aligned} \tag{3.3}$$

Denoting the approximation to $u_i(t_n)$ by $u_{i,n}$ and the approximation to $a_i(t_n)$ by $a_{i,n}$, then the second order exponential time differencing Runge-Kutta semi-discrete scheme is given by,

$$u_{i,n+1} = a_{i,n} + \frac{1}{k} A_i^2 (e^{-kA_i} - I + kA_i)[F_i(a_{1,n}, \ldots, a_{m,n}) - F_i(u_{1,n}, \ldots, u_{m,n})] \tag{3.4}$$

for $i = 1, \ldots, m$, where

$$a_{i,n} = e^{-kA_i} u_{i,n} - A_i^{-1}(e^{-kA_i} - I) F_i(u_{1,n}, \ldots, u_{m,n}). \tag{3.5}$$

Now the computational challenge is how to efficiently compute the terms $\frac{1}{k} A_i^2 (e^{-kA_i} - I + kA_i)$ and $-A_i^{-1}(e^{-kA_i} - I)$ in (3.4)-(3.5). As noted in [17], some works in the literature leave the computation to standard software at the time of implementation (see [17] and the references therein). Instead, [17] suggests a different approach which we will follow next. To achieve a second-order spatial accuracy, we use the $(1,1)$-Padé scheme to approximate e^{-kA_i}, namely

$$e^{-kA_i} \approx (2I - kA_i)(2I + kA_i)^{-1}, \quad i = 1, \ldots, m, \tag{3.6}$$

which is commonly called the Crank-Nicolson (CN) scheme in the literature. Using (3.6), we have,

$$\begin{aligned} \frac{1}{k} A_i^2 (e^{-kA_i} - I + kA_i) &\approx \frac{1}{k} A_i^2 ((2I - kA_i)(2I + kA_i)^{-1} - I + kA_i) \\ &= k(2I + kA_i)^{-1}, \qquad i = 1, \ldots, m, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} -A_i^{-1}(e^{-kA_i} - I) &\approx -A_i^{-1}((2I - kA_i)(2I + kA_i)^{-1} - I) \\ &= 2k(2I + kA_i)^{-1}, \qquad i = 1, \ldots, m. \end{aligned} \tag{3.8}$$

Then we have the following ETD-CN scheme for (2.8), for $i = 1, \ldots, m$,

$$\begin{aligned} v_{i,n+1} &= b_{i,n} + k(2I + kA_i)^{-1}[F_i(b_{1,n}, \ldots, b_{m,n}) - F_i(v_{1,n}, \ldots, v_{m,n})], \\ b_{i,n} &= (2I - kA_i)(2I + kA_i)^{-1} v_{i,n} + 2k(2I + kA_i)^{-1} F_i(v_{1,n}, \ldots, v_{m,n}), \end{aligned} \tag{3.9}$$

where we use $v_{i,n}$ and $b_{i,n}$ for $u_{i,n}$ and $a_{i,n}$, respectively in order to distinguish the semi-discrete case given by (3.4)-(3.5) from the full-discrete case (3.9) in which e^{-kA_i} is replaced by the second order Padé approximation $(2I - kA_i)(2I + kA_i)^{-1}$.

Next, we present the convergence result for the ETD-CN scheme (3.9). Let $u(t_n)$, v_n denote the vectors of the solutions $\{u_i(t_n)\}_{i=1}^m$, $\{v_{i,n}\}_{i=1}^m$, respectively. Similarly, let F denote the vector of the functions $\{F_i\}_{i=1}^m$. Let X be an appropriate finite dimensional subspace of $L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$. To show that the ETD-CN scheme (3.9) converges with second-order accuracy, we need to assume that $F(t, u(t))$ is Lipschitz on $[0, T] \times X$, i.e. it satisfies the following assumption:

Assumption 3.1. *Let $F : [0, T] \times X \to X$ and U be an open subset of $[0, T] \times X$. For every $(t, x) \in U$ there exists a neighborhood $V \subset U$ and a real number L_T such that*

$$\|F(t_1, x_1) - F(t_2, x_2)\|_X \leq L_T \Big(|t_1 - t_2| + \|x_1 - x_2\|_X \Big) \tag{3.10}$$

for all $(t_1, x_1), (t_2, x_2) \in V$.

For the convergence theorem below it is sufficient that (3.10) holds in a strip along the exact solution.

Theorem 3.1. *If Assumption 3.1 is satisfied, then, for the numerical solution the following error bound holds if* $F^{(2)} \in L^1([0,T];X)$,

$$\begin{aligned}\|u(t_n) - v_n\|_X \;\leq\; & Ck^2 \max\Big(\sup_{0\leq\tau\leq T} \|F^{(2)}(\tau, u(\tau))\|_X, \\ & \|u_0\|_X, \|Au_0\|_X\Big) + Ck^3 \sum_{j=0}^{n-1} \|AF(t_j, u_j)\|_X + Ck^2\end{aligned}$$

uniformly on $0 \leq t_n \leq T$*. The constant* C *depends on* T*, but is independent of* n *and* k*.*

The proof of Theorem 3.1 is similar to the proof of Theorem 4.7 in [17] except that no initial smoothing steps are applied and is therefore omitted.

4 POSITIVITY CONSTRAINT

It is well-known that a critical property for American option values is the positivity constraint, that is, because of the early exercise feature of American options, the value functions V_i introduced in (2.2) must satisfy the condition

$$V_i(S,t) \geq \max\{K - S, 0\},\ S \geq 0,\ 0 \leq t \leq T,\ 1 \leq i \leq m. \tag{4.1}$$

In this section we will determine an upper bound condition for the time step size k and prove that under this condition, the approximated option values generated by the ETD-CN scheme (3.9) satisfy a discrete version of the positivity constraint (4.1). See (4.16) below.

To proceed, we first rewrite (3.9) in a different form. Note that

$$(2I - kA_i)(2I + kA_i)^{-1} = 4(2I + kA_i)^{-1} - I.$$

Then the equation for $b_{i,n}$ in (3.9) can be written as

$$b_{i,n} + v_{i,n} = 4(2I + kA_i)^{-1} v_{i,n} + 2k(2I + kA_i)^{-1} F_i(v_{1,n}, \ldots, v_{m,n}),$$

or equivalently,

$$(2I + kA_i)[b_{i,n} + v_{i,n}] = 4v_{i,n} + 2kF_i(v_{1,n}, \ldots, v_{m,n}). \tag{4.2}$$

Left multiplying the equation for $v_{i,n+1}$ in (3.9) with $(2I + kA_i)$, we have,

$$\begin{aligned}(2I + kA_i)v_{i,n+1} &= (2I + kA_i)b_{i,n} + k[F_i(b_{1,n}, \dots, b_{m,n}) - F_i(v_{1,n}, \dots, v_{m,n})] \\ &= 4v_{i,n} + 2kF_i(v_{1,n}, \dots, v_{m,n}) - (2I + kA_i)v_{i,n} \\ &\quad + k[F_i(b_{1,n}, \dots, b_{m,n}) - F_i(v_{1,n}, \dots, v_{m,n})] \\ &= (2I - kA_i)v_{i,n} + k[F_i(b_{1,n}, \dots, b_{m,n}) + F_i(v_{1,n}, \dots, v_{m,n})].\end{aligned} \tag{4.3}$$

We now discretize the spatial variable S over the interval $[0, S_\infty]$. Given a positive integer M, let $h = \frac{S_\infty}{M}$ be the spatial step size. Let $V_i^{j,n}$ and $b_i^{j,n}$ denote the ETD-CN approximations $v_{i,n}$ and $b_{i,n}$ at $S_j = jh$, respectively for $0 \le j \le M$. In discretizing the differential operators A_i, $i = 1, \dots, m$, we use central differencing for the second order derivative $\frac{\partial^2}{\partial S^2}$, and use central differencing as much as possible for the derivative $\frac{\partial}{\partial S}$ to ensure positive coefficients. Note that a forward differencing may be applied to the derivative $\frac{\partial}{\partial S}$ at small j values to ensure positive coefficients. This leads to the following discrete approximation equations for the ETD-CN scheme (4.2)-(4.3). For $1 \le i \le m$, $1 \le j \le M-1$, $0 \le n < N$,

$$\begin{aligned}(2 + kD_i^j)b_i^{j,n} &= -kL_i^j b_i^{j-1,n} - kR_i^j b_i^{j+1,n} - kL_i^j V_i^{j-1,n} + (2 - kD_i^j)V_i^{j,n} - kR_i^j V_i^{j+1,n} \\ &+ 2k\sum_{l \ne i} q_{il} V_l^{j,n} + \frac{2k\varepsilon C}{V_i^{j,n} + \varepsilon - q(S_j)},\end{aligned} \tag{4.4}$$

$$\begin{aligned}(2 + kD_i^j)V_i^{j,n+1} &= -kL_i^j V_i^{j-1,n+1} - kR_i^j V_i^{j+1,n+1} \\ &\quad - kL_i^j V_i^{j-1,n} + (2 - kD_i^j)V_i^{j,n} - kR_i^j V_i^{j+1,n} \\ &\quad + k\sum_{l \ne i} q_{il} V_l^{j,n} + k\sum_{l \ne i} q_{il} b_l^{j,n} + \frac{k\varepsilon C}{V_i^{j,n} + \varepsilon - q(S_j)} + \frac{k\varepsilon C}{b_i^{j,n} + \varepsilon - q(S_j)},\end{aligned} \tag{4.5}$$

where

$$L_i^j = \begin{cases} \frac{1}{2}r_i j - \frac{1}{2}\sigma_i^2 j^2, & \text{if } r_i \le \sigma_i^2 j, \\ -\frac{1}{2}\sigma_i^2 j^2, & \text{if } r_i > \sigma_i^2 j, \end{cases} \tag{4.6}$$

$$D_i^j = \begin{cases} \sigma_i^2 j^2 + r_i - q_{ii}, & \text{if } r_i \le \sigma_i^2 j, \\ \sigma_i^2 j^2 + r_i - q_{ii} + r_i j, & \text{if } r_i > \sigma_i^2 j, \end{cases} \tag{4.7}$$

$$R_i^j = \begin{cases} -\frac{1}{2}r_i j - \frac{1}{2}\sigma_i^2 j^2, & \text{if } r_i \le \sigma_i^2 j, \\ -\frac{1}{2}\sigma_i^2 j^2 - r_i j, & \text{if } r_i > \sigma_i^2 j. \end{cases} \tag{4.8}$$

Let $U_i^{j,n} = V_i^{j,n} - q(S_j)$ and $\tilde{b}_i^{j,n} = b_i^{j,n} - q(S_j)$, $0 \le j \le M,\ 0 \le n \le N,\ 1 \le i \le m$. Then (4.4) and (4.5) are respectively transformed into:

$$\begin{aligned}(2 + kD_i^j)\tilde{b}_i^{j,n} &= -kL_i^j\tilde{b}_i^{j-1,n} - kR_i^j\tilde{b}_i^{j+1,n} - kL_i^jU_i^{j-1,n} + (2 - kD_i^j)U_i^{j,n} - kR_i^jU_i^{j+1,n} \\ &+ 2k\sum_{l\ne i} q_{il}U_l^{j,n} + \frac{2k\varepsilon C}{U_i^{j,n} + \varepsilon} - 2kr_iK,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}(2 + kD_i^j)U_i^{j,n+1} &= -kL_i^jU_i^{j-1,n+1} - kR_i^jU_i^{j+1,n+1} \\ &\quad -kL_i^jU_i^{j-1,n} + (2 - kD_i^j)U_i^{j,n} - kR_i^jU_i^{j+1,n} \\ &\quad +k\sum_{l\ne i} q_{il}U_l^{j,n} + k\sum_{l\ne i} q_{il}\tilde{b}_l^{j,n} + \frac{k\varepsilon C}{U_i^{j,n} + \varepsilon} + \frac{k\varepsilon C}{\tilde{b}_i^{j,n} + \varepsilon} - 2kr_iK.\end{aligned} \tag{4.10}$$

Define $\tilde{b}^n = \min_{i,j} \tilde{b}_i^{j,n}$ and let (i_0, j_0) be a pair of indices such that $\tilde{b}_{i_0}^{j_0,n} = \tilde{b}^n$. It then follows from (4.9) that

$$\begin{aligned}(2 + kD_{i_0}^{j_0})\tilde{b}^n &\ge -kL_{i_0}^{j_0}\tilde{b}^n - kR_{i_0}^{j_0}\tilde{b}^n - kL_{i_0}^{j_0}U_{i_0}^{j_0-1,n} + (2 - kD_{i_0}^{j_0})U_{i_0}^{j_0,n} - kR_{i_0}^{j_0}U_{i_0}^{j_0+1,n} \\ &+ 2k\sum_{l\ne i} q_{i_0l}U_l^{j_0,n} + \frac{2k\varepsilon C}{U_{i_0}^{j_0,n} + \varepsilon} - 2kr_{i_0}K.\end{aligned} \tag{4.11}$$

Using (4.6)-(4.8), we have

$$[2 + k(r_{i_0} - q_{i_0i_0})]\tilde{b}^n \ge \frac{1}{U_{i_0}^{j_0,n} + \varepsilon}\left[\Phi_{i_0}^{j_0}(U_{i_0}^{j_0-1,n}, U_{i_0}^{j_0,n}, U_{i_0}^{j_0+1,n}) + 2k\varepsilon C\right] + 2k\sum_{l\ne i_0} q_{i_0l}U_l^{j_0,n}, \tag{4.12}$$

where the function Φ_i^j is defined as

$$\Phi_i^j(u^-, u, u^+) = \left(-kL_i^ju^- + (2 - kD_i^j)u - kR_i^ju^+ - 2kr_iK\right)(u + \varepsilon). \tag{4.13}$$

Similarly, we define $U^{n+1} = \min_{i,j} U_i^{j,n+1}$ and let (i_1, j_1) be a pair of indices such that $U_{i_1}^{j_1,n+1} = U^{n+1}$. Then it follows from (4.10) that

$$\begin{aligned}[2 + k(r_{i_1} - q_{i_1 i_1})]U^{n+1} \;&\geq\; \frac{1}{U_{i_1}^{j_1,n} + \varepsilon}\,[\Phi_{i_1}^{j_1}(U_{i_1}^{j_1-1,n}, U_{i_1}^{j_1,n}, U_{i_1}^{j_1+1,n}) + k\varepsilon C] + k\sum_{l\neq i_1} q_{i_1 l} U_l^{j_1,n} \\ &\quad + k\sum_{l\neq i_1} q_{i_1 l}\tilde{b}_l^{j_1,n} + \frac{k\varepsilon C}{\tilde{b}_{i_1}^{j_1,n} + \varepsilon}.\end{aligned} \tag{4.14}$$

Lemma 4.1. *For all $1 \leq i \leq m$, $1 \leq j \leq M-1$, $u^-, u, u^+ \geq 0$, the partial derivatives satisfy $\frac{\partial \Phi_i^j}{\partial u^-} \geq 0$, $\frac{\partial \Phi_i^j}{\partial u^+} \geq 0$. Moreover, $\frac{\partial \Phi_i^j}{\partial u} \geq 0$ if the time step size k satisfies the condition*

$$k \leq \frac{2h^2}{\sigma_L^2 S_\infty^2 + r_L S_\infty h + \bar{r}_L h^2 + \frac{2r_L K h^2}{\varepsilon}}\,, \tag{4.15}$$

where $\sigma_L = \max_i \sigma_i$, $r_L = \max_i r_i$, and $\bar{r}_L = \max_i (r_i - q_{ii})$.

Proof. It is readily seen from the definition (4.13) that, for $1 \leq i \leq m$, $1 \leq j \leq M-1$, and for $u \geq 0$,

$$\frac{\partial \Phi_i^j}{\partial u^-} = -kL_i^j(u+\epsilon) \geq 0, \quad \frac{\partial \Phi_i^j}{\partial u^+} = -kR_i^j(u+\epsilon) \geq 0,$$

since $L_i^j \leq 0$ and $R_i^j \leq 0$.

Next, we have,

$$\frac{\partial \Phi_i^j}{\partial u} = -kL_i^j u^- - kR_i^j u^+ + \left[2 - kD_i^j - \frac{2kr_iK}{2u+\varepsilon}\right](2u+\epsilon).$$

Using (4.6)-(4.8), consequently, $\frac{\partial \Phi_i^j}{\partial u} \geq 0$ if

$$0 \leq 2 - kD_i^j - \frac{2kr_iK}{2u+\varepsilon} = \begin{cases} 2 - k(\sigma_i^2 j^2 + r_i - q_{ii}) - \dfrac{2kr_iK}{2u+\varepsilon}, & \text{if } r_i \leq \sigma_i^2 j, \\ 2 - k(\sigma_i^2 j^2 + r_i - q_{ii} + r_i j) - \dfrac{2kr_iK}{2u+\varepsilon}, & \text{if } r_i > \sigma_i^2 j, \end{cases}$$

which is assured if k satisfies (4.15). $\square$

Theorem 4.2. *Suppose that $C \geq 2r_L K$, $S_\infty \geq K$, and k satisfies (4.15). Then the approximate values $\{V_i^{j,n}\}$ generated by the ETD-CN scheme (4.4)-(4.5) satisfy a discrete version of the condition (4.1), i.e.,*

$$V_i^{j,n} \geq \max\{K - S_j, 0\},\; 0 \leq j \leq M,\; 0 \leq n \leq N,\; 1 \leq i \leq m. \tag{4.16}$$

Proof. First, by definition, $V_i^{j,0} = \max\{K - S_j, 0\}$, for $j = 0, \dots, M$, $i = 1, \dots, m$. Thus (4.16) holds for $n = 0$. In addition, $V_i^{0,n} = b_i^{0,n} = \max\{K - S_0, 0\} = K$ and $V_i^{M,n} = b_i^{M,n} = \max\{K - S_M, 0\} = 0$, for $n = 0, \dots, N$, $i = 1, \dots, m$, provided that $S_M = S_\infty \geq K$.

Next, we show by induction that if (4.16) holds for n, then it also holds for $n+1$, that is, we will prove that

$$V_i^{j,n+1} \geq \max\{K - S_j, 0\}, \ \forall j, i. \tag{4.17}$$

To this end, we first show that the positivity constraint holds for $b_i^{j,n}$, that is,

$$b_i^{j,n} \geq \max\{K - S_j, 0\}, \ \forall j, i. \tag{4.18}$$

We finish the proof in two steps: first we show that $b_i^{j,n} \geq K - S_j$, $\forall j, i$ and then we show that $b_i^{j,n} \geq 0 \ \forall j, i$. Using the induction hypothesis $V_i^{j,n} \geq K - S_j = q(S_j) \ \forall j, i$, we have $U_i^{j,n} = V_i^{j,n} - q(S_j) \geq 0$, $\forall j, i$. Hence, in view of Lemma 4.1, we obtain from (4.12) and (4.13) that

$$\begin{aligned}
[2 + k(r_{i_0} - q_{i_0 i_0})]\tilde{b}^n \ &\geq \ \frac{1}{U_{i_0}^{j_0,n} + \varepsilon} \left[\Phi_{i_0}^{j_0}(0,0,0) + 2k\varepsilon C\right] + 2k \sum_{l \neq i_0} q_{i_0 l} U_l^{j_0,n} \\
&\geq \ \frac{1}{U_{i_0}^{j_0,n} + \varepsilon} \left[-2k r_{i_0} K \varepsilon + 2k\varepsilon C\right] \geq 0,
\end{aligned}$$

provided that $C \geq r_i K$ for $\forall i$. Consequently, we have $\tilde{b}^n \geq 0$ which implies $\tilde{b}_i^{j,n} \geq 0$, $\forall j, i$, or equivalently, $b_i^{j,n} \geq K - S_j \ \forall j, i$.

We now show that $b_i^{j,n} \geq 0 \ \forall j, i$. Define $b^n = \min_{i,j} b_i^{j,n}$ and let (i_0, j_0) be a pair of indices such that $b_{i_0}^{j_0,n} = b^n$. Substituting i_0 and j_0 in (4.4) results in

$$\begin{aligned}
(2 + kD_{i_0}^{j_0})b^n \ &\geq \ -kL_{i_0}^{j_0} b^n - kR_{i_0}^{j_0} b^n - kL_{i_0}^{j_0} V_{i_0}^{j_0-1,n} + (2 - kD_{i_0}^{j_0}) V_{i_0}^{j_0,n} - kR_{i_0}^{j_0} V_{i_0}^{j_0+1,n} \\
&+ \ 2k \sum_{l \neq i_0} q_{i_0 l} V_l^{j_0,n} + \frac{2k\varepsilon C}{V_{i_0}^{j_0,n} + \varepsilon - q(S_{j_0})},
\end{aligned}$$

provided that k satisfies (4.15). It then follows that

$$\begin{aligned}
[2 + k(r_{i_0} - q_{i_0 i_0})]b^n \ &\geq \ -kL_{i_0}^{j_0} V_{i_0}^{j_0-1,n} + (2 - kD_{i_0}^{j_0}) V_{i_0}^{j_0,n} - kR_{i_0}^{j_0} V_{i_0}^{j_0+1,n} \\
&+ \ 2k \sum_{l \neq i_0} q_{i_0 l} V_l^{j_0,n} + \frac{2k\varepsilon C}{V_{i_0}^{j_0,n} + \varepsilon - q(S_{j_0})} \geq 0,
\end{aligned}$$

since $V_i^{j,n} \geq \max\{q(S_j), 0\}$, $\forall j, i$, and $2 - kD_{i_0}^{j_0} \geq 0$ provided that k satisfies (4.15). Hence $b^n \geq 0$ and $b_i^{j,n} \geq b^n \geq 0$, $\forall j, i$, provided that k satisfies (4.15).

Having shown that (4.18) holds, using the induction hypothesis $V_i^{j,n} \geq K - S_j = q(S_j)\ \forall j, i$, we next prove (4.17). Similar to the proof for $b_i^{j,n}$, in view of Lemma 4.1, we obtain from (4.13) and (4.14) that

$$\begin{aligned}[2 + k(r_{i_1} - q_{i_1 i_1})]U^{n+1} &\geq \frac{1}{U_{i_1}^{j_1,n} + \varepsilon}\left[\Phi_{i_1}^{j_1}(0,0,0) + k\varepsilon C\right] + k\sum_{l\neq i_1} q_{i_1 l} U_l^{j_1,n} \\ &\quad + k\sum_{l\neq i_1} q_{i_1 l}\tilde{b}_l^{j_1,n} + \frac{k\varepsilon C}{\tilde{b}_{i_1}^{j_1,n} + \varepsilon}, \\ &\geq \frac{1}{U_{i_1}^{j_1,n} + \varepsilon}\left[-2kr_{i_1}K\varepsilon + k\varepsilon C\right] \geq 0\end{aligned}$$

provided that $C \geq 2r_iK$ for $\forall i$. Consequently, we have $U^{n+1} \geq 0$ which implies $U_i^{j,n+1} \geq 0,\ \forall j, i$, or equivalently, $V_i^{j,n+1} \geq K - S_j\ \forall j, i$.

To show that $V_i^{j,n+1} \geq 0\ \forall j, i$, define $V^{n+1} = \min_{i,j} V_i^{j,n+1}$ and let (i_2, j_2) be a pair of indices such that $V_{i_2}^{j_2,n+1} = V^{n+1}$. Substituting i_2 and j_2 in (4.5) results in

$$\begin{aligned}(2 + kD_{i_2}^{j_2})V^{n+1} &\geq -kL_{i_2}^{j_2}V^{n+1} - kR_{i_2}^{j_2}V^{n+1} \\ &\quad -kL_{i_2}^{j_2}V_{i_2}^{j_2-1,n} + (2 - kD_{i_2}^{j_2})V_{i_2}^{j_2,n} - kR_{i_2}^{j_2}V_{i_2}^{j_2+1,n} \\ &\quad + k\sum_{l\neq i_2} q_{i_2 l}V_l^{j_2,n} + k\sum_{l\neq i_2} q_{i_2 l} b_l^{j_2,n} + \frac{k\varepsilon C}{V_{i_2}^{j_2,n} + \varepsilon - q(S_{j_2})} \\ &\quad + \frac{k\varepsilon C}{b_{i_2}^{j_2,n} + \varepsilon - q(S_{j_2})},\end{aligned}$$

provided that k satisfies (4.15). It then follows that

$$\begin{aligned}[2 + k(r_{i_2} - q_{i_2 i_2})]V^{n+1} &\geq -kL_{i_2}^{j_2}V_{i_2}^{j_2-1,n} + (2 - kD_{i_2}^{j_2})V_{i_2}^{j_2,n} - kR_{i_2}^{j_2}V_{i_2}^{j_2+1,n} \\ &\quad + k\sum_{l\neq i_2} q_{i_2 l}V_l^{j_2,n} + k\sum_{l\neq i_2} q_{i_2 l} b_l^{j_2,n} + \frac{k\varepsilon C}{V_{i_2}^{j_2,n} + \varepsilon - q(S_{j_2})} \\ &\quad + \frac{k\varepsilon C}{b_{i_2}^{j_2,n} + \varepsilon - q(S_{j_2})} \geq 0,\end{aligned}$$

since $V_i^{j,n} \geq \max\{q(S_j), 0\}$, $b_i^{j,n} \geq \max\{q(S_j), 0\}$, $\forall j, i$, and $2 - kD_{i_2}^{j_2} \geq 0$, provided that k satisfies (4.15). Hence, $V^{n+1} \geq 0$ and $V_i^{j,n+1} \geq V^{n+1} \geq 0,\ \forall j, i$, provided that k satisfies (4.15).

This completes the proof of the Theorem. $\square$

5 NUMERICAL EXPERIMENTS

In this section we provide two numerical examples to illustrate the performance of the ETD-CN scheme for pricing American put options in the regime-switching model.

We report the results and numerically compare them with two other methods. First we compare the ETD-CN scheme with the linearly implicit penalty method scheme developed in [14]. Secondly, binomial tree models have been a popular approach for option pricing in computational finance. Liu [18] develops an efficient tree approach for the regime-switching model that grows only linearly as the number of time steps increases. We use the tree method to calculate the approximate values of the same options and compare the ETD-CN scheme with the tree method.

Example 1. In the first example we consider the same two-regime model considered in [14]. The model parameters are set to be $q_{11} = -6$, $q_{12} = 6$, $q_{21} = 9$, $q_{22} = -9$, $r_1 = 0.1$, $r_2 = 0.05$, $\sigma_1 = 0.8$, $\sigma_2 = 0.3$. All options have maturity $T = 1$ year and exercise price $K = 9$. For the penalty terms (see (2.3)), we choose $\varepsilon = 0.001$ and $C = 1$. We use $S_\infty = 40$ for the upper bound of the asset price. The ETD-CN scheme is employed to compute the option prices in the two different regimes for a range of initial stock price S, varying from deep in-the-money, to at-the-money and to deep out-of-money options. Table I displays the results under the columns labeled 'ETD-CN' for a set of 10 representative options. These numbers are calculated by using a spatial step size $h = 0.02$ and a time step size $k = 0.0005$. For comparison, we also report the approximation prices obtained by using the linearly implicit penalty scheme (under the columns labeled 'IPS') and by using the binomial tree approach (under the columns labeled 'Tree'), which are taken from Table 1 of [14]. We see from Table I that the numbers obtained from ETD-CN scheme are very close to those obtained from the linearly implicit penalty scheme in [14]. They are both comparably close to the approximation prices obtained from the binomial tree approach. However as noted in Section 3, the ETD-CN scheme achieves a second order convergence rate, a significant improvement over the linearly implicit penalty scheme. The second order convergence rate is also numerically illustrated in Table II below.

Figure 1 displays the American option prices as a function of the initial stock price S from $S = 0$ to $S = 20$ at time $t = 0$, obtained using the ETD-CN scheme. Figure 2 displays the price surfaces as a function of both S and t over the rectangular domain $[0, 20] \times [0, 1]$.

To numerically show that the proposed ETD-CN scheme is second order accurate, we compute the value of the at-the-money option ($S = K = 9$) using different values

TABLE I. Comparison of American put option prices in a two-regime model.

S	ETD-CN $\alpha_0 = 1$	IPS	Tree	ETD-CN $\alpha_0 = 2$	IPS	Tree
3.5	5.5001	5.5001	5.5000	5.5012	5.5012	5.5000
4.0	5.0067	5.0067	5.0031	5.0016	5.0016	5.0000
4.5	4.5485	4.5486	4.5432	4.5192	4.5194	4.5117
6.0	3.4196	3.4198	3.4144	3.3563	3.3565	3.3503
7.5	2.5886	2.5887	2.5844	2.5077	2.5078	2.5028
8.5	2.1597	2.1598	2.1560	2.0721	2.0722	2.0678
9.0	1.9756	1.9756	1.9722	1.8859	1.8860	1.8819
9.5	1.8089	1.8090	1.8058	1.7181	1.7181	1.7143
10.5	1.5213	1.5214	1.5186	1.4301	1.4301	1.4267
12.0	1.1825	1.1827	1.1803	1.0945	1.0945	1.0916

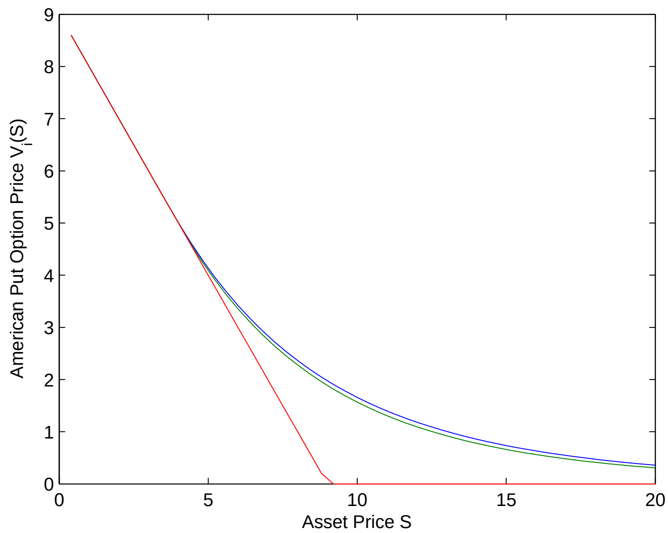

FIG. 1. American put option price curves at $t = 0$ using the ETD-CN scheme (Two regimes, upper curve: $\alpha_0 = 1$, middle curve: $\alpha_0 = 2$, bottom curve: payoff function.)

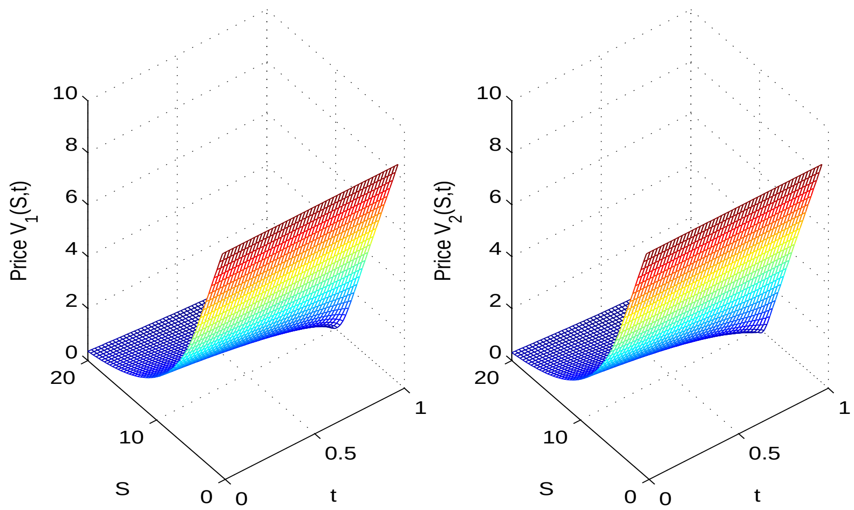

FIG. 2. American put option price surfaces using the ETD-CN scheme (Two regimes, left surface: $\alpha_0 = 1$, right surface: $\alpha_0 = 2$.)

for the spatial step size h and the time step size k. We then calculate the order of the ratio of successive changes of option values as the grid is refined. These results are reported in Table II. We clearly see that the rate of convergence is second order. **Note that in Table II, the rate is calculated by the formula** $\log_2(E_1/E_2)$ **where** E_1 **and** E_2 **are the successive changes of option values. For example, in calculating the first rate (**1.9985 **for regime 1 and** 1.9986 **for regime 2),** E_1 **is the difference between the first and second option price,** E_2 **is the difference between the second and third option price.**

Example 2. We further test the ETD-CN scheme using a four-regime model. The state space of the Markov chain α_t is $\mathcal{M} = \{1, 2, 3, 4\}$ and the generator is specified as

$$Q = \begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}.$$

TABLE II. American option price using ETD-CN scheme and rate of convergence.

h	k	$\alpha_0 = 1$	Rate	$\alpha_0 = 2$	Rate
0.2	0.002	1.97524414		1.88560720	
0.1	0.001	1.97548070	N/A	1.88586316	N/A
0.05	0.0005	1.97553990	1.9985	1.88592721	1.9986
0.025	0.00025	1.97555469	2.0010	1.88594323	1.9993
0.0125	0.000125	1.97555838	2.0029	1.88594722	2.0054
0.00625	0.0000625	1.97555930	2.0039	1.88594821	2.0109

Thus the market can be in any of the four regimes with equal probability. The model parameters are chosen as

$$\sigma_1 = 0.9, \quad \sigma_2 = 0.5, \quad \sigma_3 = 0.7, \quad \sigma_4 = 0.2,$$

$$r_1 = 0.02, \quad r_2 = 0.1, \quad r_3 = 0.06, \quad r_4 = 0.15.$$

We use $S_\infty = 100$, $h = 0.04$, $k = 0.0005$, $\varepsilon = 0.001$, and $C = 1.5$ in the implementation of the ETD-CN scheme. Note that the same values for those parameters are used in [14] for implementing the linearly implicit penalty scheme. In Table III we report the approximate prices of six options in the four different regimes, obtained by using the ETD-CN approximation scheme, the linearly implicit penalty scheme, and the binomial tree approach. All of those options have the same maturity $T = 1$ year and the same exercise price $K = 9$, while the initial stock price changes from $S_0 = 4.5$ to $S_0 = 12$. We observe again that the three methods produce very close approximate option prices.

Figure 3 displays the American option prices as a function of the initial stock price S from $S = 0$ to $S = 20$ at time $t = 0$, obtained using the ETD-CN scheme. Figure 4 displays the price surfaces as a function of both S and t over the rectangular domain $[0, 20] \times [0, 1]$.

TABLE III. Comparison of American put option prices in a four-regime model.

Regime	Method	$S_0 = 4.0$	$S_0 = 6.0$	$S_0 = 7.5$	$S_0 = 9.0$	$S_0 = 10.5$	$S_0 = 12.0$
$\alpha_0 = 1$	ETD-CN	5.2611	3.9141	3.1513	2.5641	2.1113	1.7578
	IPS	5.2610	3.9140	3.1512	2.5642	2.1117	1.7588
	Tree	5.2484	3.9044	3.1433	2.5576	2.1064	1.7545
$\alpha_0 = 2$	ETD-CN	5.0009	3.1812	2.2384	1.5884	1.1451	0.8404
	IPS	5.0009	3.1814	2.2387	1.5886	1.1452	0.8404
	Tree	5.0000	3.1732	2.2319	1.5834	1.1417	0.8377
$\alpha_0 = 3$	ETD-CN	5.0443	3.5173	2.6813	2.0623	1.6057	1.2658
	IPS	5.0443	3.5173	2.6813	2.0622	1.6057	1.2658
	Tree	5.0348	3.5092	2.6746	2.0568	1.6014	1.2625
$\alpha_0 = 4$	ETD-CN	5.0002	3.0008	1.6664	0.9903	0.6580	0.4725
	IPS	5.0002	3.0008	1.6676	0.9911	0.6583	0.4725
	Tree	5.0000	3.0000	1.6574	0.9855	0.6553	0.4708

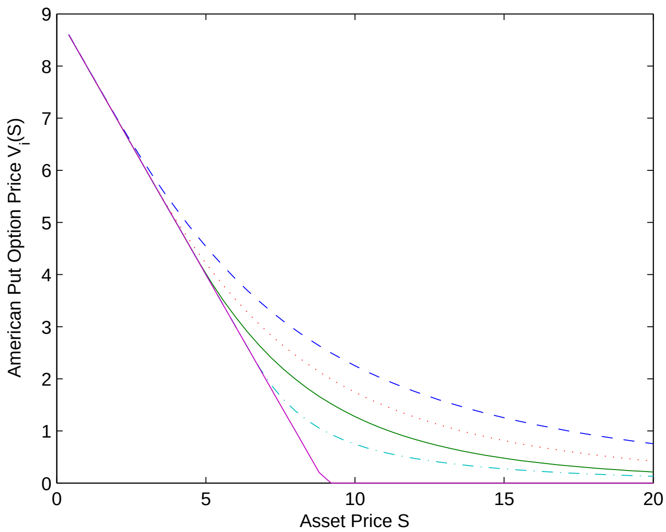

FIG. 3. American put option price curve at $t = 0$ using the ETD-CN scheme (Four regimes. The five curves (from top to bottom) correspond to regime $\alpha_0 = 1, 3, 2, 4$, and the option payoff function, respectively.)

6 CONCLUDING REMARKS

We develop a new numerical scheme for solving a class of complex PDE systems arising in the American option pricing problem in regime-switching models. This scheme utilizes the penalty method approach and an efficient exponential time differencing Crank-Nicolson (ETD-CN) method, resulting in a fast numerical scheme. We numerically compare the ETD-CN scheme with two other schemes, namely a linearly implicit penalty method scheme and a binomial tree method. Numerical results illustrate the second order convergence of the ETD-CN scheme. In addition, we establish an upper bound condition for the time step size and prove that under this condition the ETD-CN scheme satisfies a discrete version of the positivity constraint for American option values. An interesting topic for future research will be to extend the ETD-CN method to multi-asset American option pricing problems in the regime-switching models. **Another interesting topic will be to develop efficient Monte-Carlo simulation method for pricing American options with regime-switching and compare it with the ETD-CN method.**

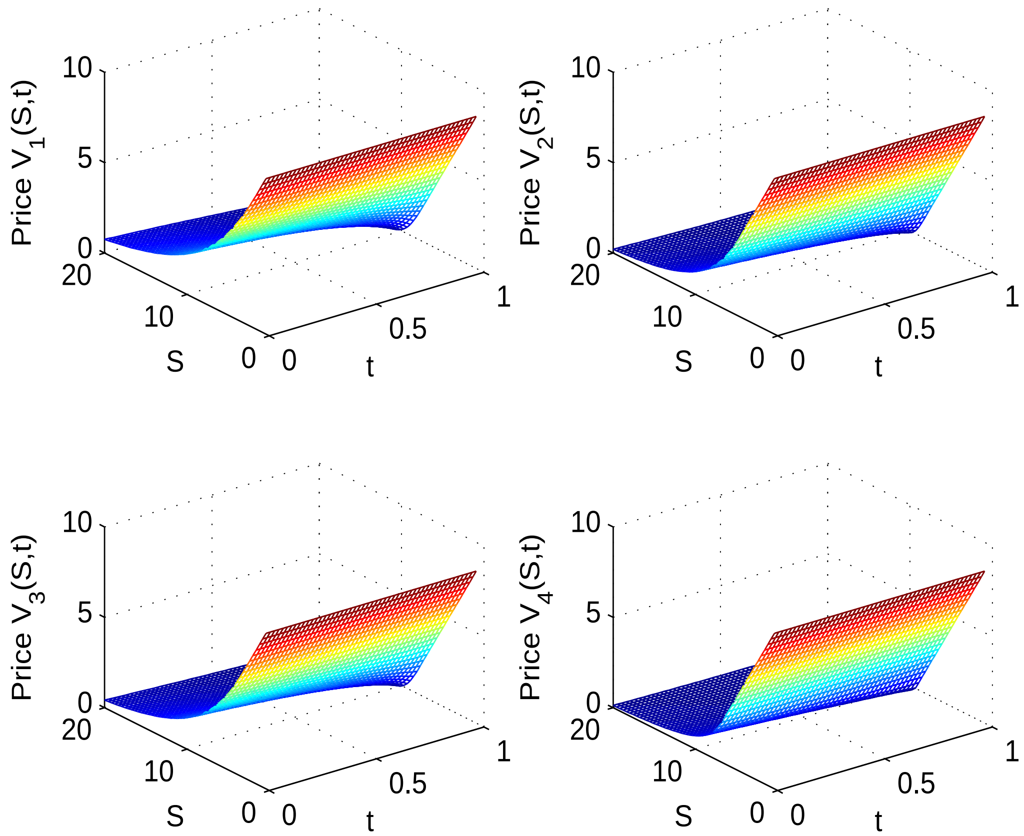

FIG. 4. American put option price surfaces using the ETD-CN scheme (Four regimes).

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